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GENERAL THEOREMS ON MULTIPLE INTEGRALS.

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In Liouville's theorem for the reduction of a certain class of definite multiple integrals, the integrations comprise all positive values of the variables which do not transgress a limiting inequality, which either is of, or may easily be reduced to, a linear form. Take for illustration the case of two variables, and let $mx + ny < h$ be the limiting inequality in question, m, n and h being positive. Then, geometrically, $mx + ny = h$ is the equation of a straight line which forms the base of a triangle of which the intercepts of the positive half axes of co-ordinates are the sides, and our integration extends over the whole surface of this triangle. A similar interpretation may of course be given in the case of three variables. But to return to that of two. Let $mx + ny = h$ cut the axis of x in the point M and that of y in the point N; conceive another straight line $m'x + n'y = h'$; m', n', h' being also all positive; and let it cut the axes in M', N' respectively. Let us suppose for distinctness that $\frac{m}{m'}$ is greater than $\frac{n}{n'}$. Then, if the value of $\frac{h}{h'}$ be intermediate between those of the two fractions $\frac{m}{m'}$ and $\frac{n}{n'}$, it will be easily seen that the two lines must intersect in some point A, lying in the positive quadrant of co-ordinates, and that we shall have a quadrilateral $OMAN'$, (O being the origin of co-ordinates,) formed by the axes and by the two bounding lines. If now we integrate any function of x and y for all positive values of the variables not transgressing the *two* inequalities $mx + ny \leq h, m'x + n'y \leq h'$,

we shall in effect integrate over the surface of the quadrilateral $OMAN'$. But if the two lines did not intersect within the positive quadrant, then one or other bounding inequality would be inoperative, and we should in effect integrate over the surface, not of a quadrilateral, but of a triangle, as in the case contemplated by Liouville's theorem. It is manifest that we may have, instead of two limiting inequalities, any larger number we please, and that our integrations may thus be made to extend over an irregular polygon of a greater or less number of sides. I do not believe that any writer on multiple integrals has considered the case in which the limits are given by more than one inequality, but the restriction to that of one is clearly unnecessary.

Let us suppose there are r variables $x, y \ldots . z$, and that we have to evaluate the integral

$$\int_0 dx \ldots \int_0 dz\, e^{-ax \ldots -cz}\, \phi\,(mx + \ldots pz)\, \phi_{,}\,(m_{,}x + \ldots p_{,}z) \ldots . \quad (1),$$

subject to the two inequalities

$$mx + \ldots . pz \leqq h, \qquad m_{,}x + \ldots . p_{,}z \leqq h_{,},$$

$m \ldots . p, h\,;\ m_{,} \ldots . p_{,}, h_{,}$ being all positive; and ϕ and $\phi_{,}$ any functions whose values may be represented within the limits of integration by Fourier's theorem.

Let the value of the integral in question be I; then, by considerations analogous to those of which I made use in a paper which appeared at the commencement of the last volume of the *Journal*, we shall have

$$I = \frac{1}{\pi^2}\int^{h} \phi u\, du \int^{h_{,}} \phi_{,}u_{,}\, du_{,} \int_0^\infty da \int_0^\infty da_{,} . G,$$

where

$$G = \int_0^\infty dx \ldots \int_0^\infty dz\, e^{-ax \ldots -cz} \cos a(mx + \ldots pz - u) \cos a_{,}(m_{,}x + \ldots p_{,}z - u_{,}),$$

and the lower limits of integration with respect to u and u' may be any negative quantities.

I remark, in the first place, that

$$\int_0^\infty da \int_0^\infty da_1\, G = \tfrac{1}{4}\int_{-\infty}^\infty da \int_{-\infty}^\infty da_{,} H, \quad \text{where}$$

$$H = \int_0^\infty dx \ldots \int_0^\infty dz\, e^{-ax \ldots -cz} \cos\{(am + a_{,}m_{,})x + \ldots (ap + a_{,}p_{,})z - au - a_{,}u_{,}\},$$

and therefore

$$I = \frac{1}{4\pi^2}\int^{h} \phi u\, du \int^{h'} \phi_{,}u_{,}\, du_{,} \int_{-\infty}^\infty da \int_{-\infty}^\infty da_{,} H.$$

Let $\quad H = K \cos(au + a_{,}u_{,}) + L \sin(au + a_{,}u_{,}).$

Then it will be easily seen that

$$K=\frac{N}{D}, \quad L=\frac{N'}{D};$$

where, if we take the case of three variables,

$$N = abc\left(1 - \frac{am + a_, m_,}{a}\,\frac{an + a_, n_,}{b} - \frac{am + a_, m_,}{a}\,\frac{ap + a_, p_,}{c} - \frac{an + a_, n_,}{b}\,\frac{ap + a_, p_,}{c}\right),$$

$$N' = abc\left(\frac{am + a_, m_,}{a} + \frac{an + a_, n_,}{b} + \frac{ap + a_, p_,}{c} - \frac{am + a_, m_,}{a}\,\frac{an + a_, n_,}{b}\,\frac{ap + a_, p_,}{c}\right).$$

$$D = \{a^2 + (am + a_, m_,)^2\}\,\{b^2 + (an + a_, n_,)^2\}\,\{c^2 + (ap + a_, p_,)^2\}.$$

(Precisely the same law of formation of these quantities would obtain if we were to take any number of variables. I have taken the case of three merely for distinctness of representation.)

Putting for cos $(au + a_, u_,)$ and sin $(au + a_, u_,)$ their exponential values, we find that

$$HD = a\dots c\left\{1 - \sqrt{(-1)}\,\frac{am + a_, m_,}{a}\right\}\dots\left\{1 - \sqrt{(-1)}\,\frac{ap + a_, p_,}{c}\right\}e^{(\alpha u + \alpha_, u_,)\sqrt{(-1)}}$$
$$+\, a\dots c\left\{1 + \sqrt{(-1)}\,\frac{am + a_, m_,}{a}\right\}\dots\left\{1 + \sqrt{(-1)}\,\frac{ap + a_, p_,}{c}\right\}e^{-(\alpha u + \alpha_, u_,)\sqrt{(-1)}};$$

and as

$$a^2 + (am + a_, m_,)^2 = a^2\left\{1 - \sqrt{(-1)}\,\frac{am + a_, m_,}{a}\right\}\left\{1 + \sqrt{(-1)}\,\frac{am + a_, m_,}{a}\right\},$$

$$H = \frac{e^{(\alpha u + \alpha_, u_,)\sqrt{(-1)}}}{\{a + \sqrt{(-1)}\,(am + a_, m_,)\}\dots\{c + \sqrt{(-1)}\,(ap + a_, p_,)\}} + \frac{e^{-(\alpha u + \alpha_, u_,)\sqrt{(-1)}}}{\{a - \sqrt{(-1)}\,(am + a_, m_,)\}\dots\{c - \sqrt{(-1)}\,(ap + a_, p_,)\}}.$$

Now, assume that

$$\frac{1}{(a + am + a_, m_,)\dots(c + ap + a_, p_,)}$$
$$= \frac{F_{ab}}{(a + am + a_, m_,)(b + an + a_, n_,)} + \frac{F_{ac}}{(a + am + a_, m_,)(c + ap + a_, p_,)} + \&c. \quad \dots\dots\dots (2);$$

where F_{ab}, F_{ac}, &c. are independent of a and $a_,$. This assumption is justifiable because it introduces $\frac{r.r-1}{2}$ disposable quantities F, viz. as many as there are combinations two and

two of the r quantities $a, b \ldots . c$, and it will be easily seen that there are the same number of conditions to be satisfied.

Consequently as

$$e^{(au+a'u')\sqrt{(-1)}} = \cos(au + a'u') + \sqrt{(-1)}\sin(au + a'u'),$$

we shall have

$$H = F_{ab}\left.\begin{array}{l} \{ab - (am + a_, m_,)(an + a_, n_,)\}\cos(au + a_, u_,) \\ \quad + \{a(an + a_, n_,) + b(am + a_, m_,)\}\sin(an + a_, n_,) \\ \text{divided by} \\ \quad \{a^2 + (am + a'm')^2\}\ \{b^2 + (an + a'n')^2\} \end{array}\right\} + \&c.$$

Let us next assume $u = mx + ny$, $u_, = m_, x + n_, y$, x and y being here two new variables; also $\alpha' = am + a_, m_,$, and $\beta' = an + a_, n_,$: then the coefficient of F_{ab} in the expression of H will become

$$\frac{(ab - \alpha'\beta')\cos(\alpha' x + \beta' y) + (a\beta' + b\alpha')\sin(\alpha' x + \beta' y)}{(a^2 + \alpha'^2)(b^2 + \beta'^2)}.$$

Moreover $dudu_1 dada_1$ will be replaced by $dxdyd\alpha' d\beta'$; and therefore, as we have

$$I = \frac{1}{4\pi^2}\int^h \phi u du \int^h \phi_, u_, du_, \int_{-\infty}^{+\infty} da \int_{-\infty}^{+\infty} da_,\ H, \text{ we shall have}$$

$$I = \frac{1}{4\pi^2}\,\Sigma F_{ab} \iint \phi\,(mx + ny)\,\phi_,(m_, x + n_, y)\,dxdyM,$$

where the sign of summation extends to all the quantities F, and where

$$M = \int_{-\infty}^{+\infty} d\alpha' \int_{-\infty}^{+\infty} d\beta' \frac{(ab - \alpha'\beta')\cos(\alpha' x + \beta' y) + (a\beta' + b\alpha')\sin(\alpha' x + \beta' y)}{(a^2 + \alpha_,^2)(b^2 + \beta_,^2)}.$$

From the known integrals

$$\int_{-\infty}^{+\infty} \frac{\cos \alpha x . d\alpha}{a^2 + \alpha^2} = \frac{\pi}{a}\,e^{\mp ax} \int_{-\infty}^{+\infty} \frac{\alpha \sin \alpha x . d\alpha}{a^2 + \alpha^2} = \pm\,\pi e^{\mp ax},$$

the upper signs to be taken when x is positive, it follows that

$$M = \pi^2 e^{\mp ax \mp by}\,(1 \pm 1 \pm 1 \pm 1).$$

If x and y are both positive, the bracket becomes $1 + 1 + 1 + 1$ or 4; if x only be negative, it becomes $1 - 1 - 1 + 1$ or 0; if y only be negative, it becomes $1 - 1 + 1 - 1$ or 0; and similarly if both x and y are negative. Thus generally

$$M = 4\pi^2 e^{-ax-by} \text{ or } M = 0.$$

There are, indeed, exceptional cases; as if y be zero, x being positive, when $M = 2\pi^2 e^{-ax}$, and similarly if x be zero, y being

positive; and again, if x and y are both zero, when $M = \pi^2$: but of these, as we are about to multiply M by the element $dx\,dy$, it is unnecessary to take account. Therefore, in integrating for x and y, we include only positive values of the variables; and as u and u' are not to be greater than h and $h_{,}$ respectively, x and y must be such as not to transgress the inequalities

$$mx + ny \leqq h, \quad m_{,}x + n_{,}y \leqq h_{,}.$$

Thus we find that

$$I = \Sigma F_{ab} \int_0 dx \int_0 dy\, \phi(mx + ny)\, \phi_{,}(m_{,}x + n_{,}y)\, e^{-ax-by},$$

the limits being given by the two above-written inequalities. It appears, therefore, that the integral (1), when there are two limiting inequalities, is reducible to the sum of a series of double integrals.

This result is analogous to that which is obtained in the case of the function $\phi(mx + \dots pz)\, e^{-ax \dots -cz}$, in the paper already referred to.

It remains to determine the form of the quantity F_{ab}. This is done at once by multiplying equation (2) by $(a + \alpha m + \alpha_{,} m_{,})(b + \alpha n + \alpha_{,} n_{,})$, and replacing α, $\alpha_{,}$ by values which make both these factors vanish. It hence appears that

$$F_{ab} = \frac{(mn_{,} - m_{,}n)^{r-2}}{\{c(mn_{,} - m_{,}n) + a(np_{,} - n_{,}p) + b(pm_{,} - p_{,}m)\} \dots},$$

the denominator being the continued product of $r - 2$ factors, each of the same form as the one written down. Of course the other quantities F are obtained in the same manner.

Let us now take the more general case in which there are s limiting inequalities, s being less than r, and in which the function to be integrated is

$$\phi_1(m_1x + \dots p_1z) \dots \dots \phi_s(m_sx + \dots p_sz)\, e^{-ax \dots -cz},$$

the inequalities in question being

$$m_1x + \dots p_1z \leqq h_1, \dots \dots m_sx + \dots p_sz \leqq h_s.$$

We shall arrive at a perfectly analogous result in this more general case. In the first place the integral sought may be thus written,

$$\frac{1}{\pi^s} \int^{h_1} \phi_1 u_1 du_1 \dots \int^{h_s} \phi_s u_s du_s \int_0^\infty d\alpha_1 \dots \int_0^\infty d\alpha_s G, \text{ where}$$

$$G = \int_0^\infty dx \dots \int_0^\infty dz\, e^{-ax- \dots cz} \cos \alpha_1 (m_1x + \dots p_1z - u_1) \dots \cos \alpha_s (m_sx + \dots p_sz - u_s).$$

Now a little consideration will convince us that

$$\int_0^\infty da_1 \dots \int_0^\infty da_s G = \frac{1}{2^s}\int_{-\infty}^{+\infty} da_1 \dots \int_{-\infty}^{+\infty} da_s H, \text{ where}$$

$$H = \int_0^\infty dx \dots \int_0^\infty dz\, e^{-ax - \dots - cz} \cos\left[x\Sigma am + \dots + z\Sigma ap - \Sigma au\right]:$$

for if we take the expression

$$\cos\left[x\Sigma am + \dots + z\Sigma ap - \Sigma au\right],$$

make a_1 negative, add the resulting expression to the original one: then in the two terms thus got make a_2 negative, and as before add the results, we shall, continuing this process, get in all 2^s terms, which will be found to be equal to 2^s times the continued product of the cosines involved in G.

Effecting the integrations indicated in H, we see that

$$H = \frac{N\cos\Sigma au + N'\sin\Sigma au}{D},$$

where $\quad D = \{a^2 + (\Sigma am)^2\} \dots \{c^2 + (\Sigma ap)^2\},$

and N and N' follow the same law of formation as in the particular case already considered, except that for $\dfrac{am + a'm'}{a}$, &c. we substitute $\dfrac{\Sigma am}{a}$, &c. With this remark we perceive that

$$H = \frac{e^{\Sigma au\sqrt{(-1)}}}{\{a + \sqrt{(-1)}\,\Sigma am\}\dots\{c + \sqrt{(-1)}\,\Sigma ap\}} + \frac{e^{-\Sigma au\sqrt{(-1)}}}{\{a - \sqrt{(-1)}\,\Sigma am\}\dots\{c - \sqrt{(-1)}\,\Sigma ap\}}\dots(3).$$

The assumption now to be made is that

$$\frac{1}{(a + \Sigma am)\dots(c + \Sigma ap)} = \Sigma\frac{F}{\Delta}\dots\dots(2'),$$

where Δ is the product of every set of s factors taken out of the whole number of r factors

$$a + \Sigma am, \quad \dots \; c + \Sigma ap,$$

and F is independent of $a_1 \dots a_s$.

There will thus be $\dfrac{r \cdot r-1 \dots r-s+1}{1 \cdot 2 \dots s}$ disposable quantities F, which will be found to be the number required to make (2′) identically true. Consequently we shall have

$$H = \Sigma F\frac{\nu\cos\Sigma au + \nu'\sin\Sigma au}{\delta},$$

where δ is the product of s factors of the form $a^2 + (\Sigma am)^2$; and ν and ν' are formed just as in the case of $s = 2$: that is to say, we shall have

$$\nu = a \ldots c\,(1 - C_2 + C_4 - \&c.), \quad \nu' = a \ldots c\,(C_1 - C_3 + \&c.),$$

where C_t is the sum of the products of every combination that can be made of the s quantities $\dfrac{\Sigma am}{a}$, &c., taken t and t together.

In order to simplify the expression

$$\frac{\nu \cos \Sigma au + \nu' \sin \Sigma au}{\delta},$$

let us denote the s quantities Σam, &c., which are involved in it, by the single symbols $\beta_1 \ldots \beta_s$, and assume

$$u_1 = \Sigma m_1 x, \quad u_2 = \Sigma m_2 x, \quad \ldots . \; u_s = \Sigma m_s x,$$

the sign of summation Σ extending only to that set of s out of the r quantities $x \ldots . z$, which corresponds to the factors involved in the denominator δ. Of course x, y, &c. are here, as before, new variables. (In the case of $s = 3$, for instance, these assumptions will be of the form

$$\begin{aligned} \beta_1 &= a_1 m_1 + a_2 m_2 + a_3 m_3, & u_1 &= m_1 x + n_1 y + p_1 z, \\ \beta_2 &= a_1 n_1 + a_2 n_2 + a_3 n_3, & u_2 &= m_2 x + n_2 y + p_2 z, \\ \beta_3 &= a_1 p_1 + a_2 p_2 + a_3 p_3, & u_3 &= m_3 x + n_3 y + p_3 z.) \end{aligned}$$

It follows from this that $du_1 \ldots du_s\, da_1 \ldots da_s$ will be replaced by $dx \ldots dy\,.\, d\beta_1 \ldots d\beta_s$, and that the factors in δ will take the simpler form $a^2 + \beta_1^2$, $b^2 + \beta_2^2$, &c.; while Σau will become $\beta_1 x + \beta_2 y + \ldots .$

The integrations with respect to β extend, like those for a, from $-\infty$ to $+\infty$. Let

$$\int_{-\infty}^{+\infty} d\beta_1 \ldots \int_{-\infty}^{+\infty} d\beta_s \frac{\nu \cos \Sigma\beta x + \nu' \sin \Sigma\beta x}{\delta} = M'.$$

Then, from the obvious analogy between the forms ν and ν', and those of the developments of $\cos \Sigma\beta x$ and $\sin \Sigma\beta x$ respectively, it follows that if x, y, &c. are all positive,

$$M' = \pi^s e^{-ax-by \cdots}\,(1 + 1 + \ldots .),$$

there being twice as many units within the brackets as there are terms in the development of $\sin (f_1 + \ldots . f_s)$, or of $\cos (f_1 + \ldots . f_s)$, that is to say, twice 2^{s-1} or 2^s.

Moreover, if any one, as x, of the quantities x, y, &c., is negative, $M' = 0$; and this, whether it alone is negative or

any others, are so too. For if $x = -x'$, let its coefficient β_1 be assumed equal to $-\beta_1'$, when the expression of M' becomes of the same form as if x were positive, except that ν and ν' are changed by having $-\beta_1'$ wherever β_1 occurred previously. Now none of the quantities β can occur raised to any power, and therefore every term involving β_1 will change sign when β_1 is replaced by $-\beta_1'$. Hence we shall have

$$M_{,} = \pi^s e^{-ax'-by-\cdots}(1 \pm 1 \ldots .),$$

there being as many negative units as positive within the brackets, since in the development of $\sin(f_1 + \ldots . f_s)$ or $\cos(f_1 + \ldots . f_s)$ there are 2^{s-2} terms independent of the sine of f_1 and 2^{s-2} terms which involve that quantity, and which therefore change sign when f_1 does so. Hence the quantity within the bracket, and consequently $M_{,}$, is equal to zero if x be negative; and so, of course, for the other variables $y \ldots z$.

M will, in particular cases analogous to those already noticed, assume exceptional or limiting values, but of these we need not take account. And thus we arrive at the following remarkable theorem:

The definite integral of r *variables* x. . . . z

$$\int_0 dx \ldots . \int_0 dz\, \phi_1(m_1x + \ldots . p_1z) \ldots . \phi_s(m_sx + \ldots . p_sz)\, e^{-ax - \cdots cz},$$

whose limits are given by s *inequalities*

$$m_1x + \ldots . p_1z \leqq h_1, \ldots . m_sx + \ldots . p_sz \leqq h_s,$$

can generally be expressed as a linear function of

$$\frac{r.(r-1)\ldots(r-s+1)}{1.2\ldots s}$$

integrals of s *variables each. The form of each of these integrals may be deduced from the original integral by omitting from it any set of* r – s *of the variables, and similarly the form of the limiting inequalities may be got by omitting the same set of variables from the original inequalities* (r > s).

In certain cases, however, when the constants a, m, &c. have particular values, the theorem fails because the assumption (2′) becomes illegitimate. This failure is indicated by certain of the quantities F becoming infinite. To determine the form of F, we have merely to multiply (2′) by Δ, and then to equate to zero all the s factors of which Δ is composed. All the quantities F, except the particular one under consideration, will then disappear, and we have s equations determining the s quantities a. Hence it will appear that

F is equal to a fraction whose numerator is unity, and denominator equal to the value assumed by the product of the remaining $r - s$ factors, when the values already assigned for the quantities a, &c. have been substituted for them; a result which it is obvious can be immediately expressed in the notation of determinants. F will therefore become infinite if our equating the s factors by which it is divided in (2′) to zero will make one or more of the remaining $r - s$ factors vanish. Let it make t of these factors vanish; then equating these t factors also to zero, we get in all $s + t$ equations, which are equivalent to s independent ones. Therefore any set of s out of these $s + t$ equations will satisfy the remaining t equations. Hence

$$\frac{(s+t)(s+t-1)\ldots.(t+1)}{1.2.\ldots t}$$

of the quantities F will become infinite, and therefore the second side of (2′) will consist of finite terms and of a finite quantity expressed in the form of the sum of that number of infinite terms. This indetermination of course indicates a change in the form of the function, the general character of which the reader will have little difficulty in perceiving. But the consideration of these particular cases, some of which are interesting, must be deferred to another occasion.

I am inclined to believe that the processes developed in this paper will admit both of simplification and extension. For the exponential function we may substitute with certain modifications any function of $ax + \ldots cz$, in accordance with a result given by Mr. Boole in his very interesting Memoir on a new Method in Analysis, which is published in the *Transactions of the Royal Society*. (This result would include the one which I obtained in the last volume of the *Journal*, from which however it might be deduced.)

Thus, if in the theorem established in this paper we replace $a, b \ldots c$ by $ka, kb \ldots kc$, k being a wholly arbitrary quantity, we may, comparing the coefficients of its powers, deduce new theorems from the given one. Developing the first side of the equation, the coefficient of k^n will be

$$\frac{1}{1.2\ldots n}\int_0 dx \ldots \int_0 dz\, \phi_1(m_1x + \ldots p_1z)\ldots \phi_s(m_sx + \ldots p_sz)\,(ax + \ldots cz)^n,$$

and in the second it will be the sum of a series of terms of the form

$$\frac{F}{1.2\ldots(n+r-s)}\int_0 dx \int_0 dy \ldots \phi(m_1x + n_1y + \ldots)$$
$$\ldots\phi_s(m_sx + n_sy + \ldots)\,(ax + by + \ldots)^{n+r-s},$$

as it is manifest that F will become $\dfrac{F}{k^{r-s}}$. Hence, if $\psi(ax + \ldots cz)$ be such a function that its development may be substituted for it in the integrations, we shall have

$$\int_0 dx \ldots \int_0 dz\, \phi_1(m_1x + \ldots p_1z) \ldots \phi_s(m_sx + \ldots p_sz)\, \psi(ax + \ldots cz)$$
$$= \Sigma F \int_0 dx \int_0 dy \ldots \phi_1(m_1x + n_1y + \ldots) \ldots \phi_s(m_sx + n_sy + \ldots)$$
$$\psi_1(ax + by + \ldots),$$

where $\dfrac{d^{r-s}}{dt^{r-s}}\psi_1 t = \psi t$ and all the differential coefficients of $\psi_1 t$ of an order lower than the $(r - s)^{\text{th}}$ vanish for $t = 0$. This is, I believe, in the case of s equal to unity, precisely equivalent to one of Mr. Boole's results. It might also, I imagine, be obtained without having recourse to developments.

ON THE EQUATION OF LAPLACE'S FUNCTIONS.

By George Boole.

The partial differential equation of the second order, known as the Equation of Laplace's Functions, and usually expressed in the form

$$\frac{d}{d\mu}(1 - \mu^2)\frac{du}{d\mu} + \frac{1}{1 - \mu^2}\frac{d^2u}{d\phi^2} + n(n+1)\,u = 0 \ldots\ldots (1),$$

is not more remarkable for the importance of its physical applications, than for the difficulties which it presents in a purely mathematical point of view. Mr. Hargreave, in the *Philosophical Transactions* for 1841, first succeeded in obtaining an expression for the complete integral. His analysis is original and most ingenious. Assuming two new variables, x and y, connected with the former ones by the relations

$$x = \phi + \tfrac{1}{2}\sqrt{-1}\,\log\frac{1+\mu}{1-\mu}, \quad y = \phi - \tfrac{1}{2}\sqrt{-1}\,\log\frac{1+\mu}{1-\mu} \ldots (2),$$

he reduces the equation to the form

$$\frac{d^2u}{dx\,dy} + \frac{n(n+1)\,u}{4\cos^2\dfrac{x-y}{2}} = 0 \ldots\ldots\ldots\ldots (3),$$

and, by a process of reduction which it is not necessary here to explain, he ultimately finds

$$u = \ldots\int \cos^{-2}\frac{y-x}{2}\int \cos^{-2}\frac{y-x}{2}\left\{\int \cos^{2n}\frac{x-y}{2}\,\chi(y)\,dy + \psi(x)\right\} dy\,dy$$
$$\ldots\ n \text{ times} \ldots\ldots (4),$$

in which $\chi(y)$, $\psi(x)$, denote arbitrary functions of y and x.

The correctness of this solution I have since verified by a different analysis. The result is, however, so complicated by signs of integration, that the determination of the arbitrary functions is extremely difficult, and the particular deductions in Mr. Hargreave's paper are, I conceive, erroneous. Under these circumstances, it becomes an object of interest to inquire, whether the equation does not admit of a better form of solution. Such is the question which I propose to consider in this paper. I shall shew that the complete integral may be expressed in a form at once symmetrical and free from signs of integration, and shall, by a proper determination of the arbitrary functions, deduce from it the actual forms of Laplace's coefficients. As the method to be employed is not perhaps generally known, it may be proper to state first the preliminary theorems on which it depends, referring the reader, for a more particular account of them, to a paper on a "General Method in Analysis," published in the *Philosophical Transactions* for 1844, Part II.

1. It is known that the symbols $\frac{d}{d\theta}$ and ϵ^{θ}, or as for convenience we may write D and ϵ^{θ}, satisfy the following relations:

$$f(D)\,\epsilon^{m\theta} = f(m)\,\epsilon^{m\theta} \ldots\ldots\ldots\ldots (5),$$
$$f(D)\,\epsilon^{m\theta}u = \epsilon^{m\theta}f(D+m)\,u \ldots\ldots (6).$$

PROP. 1. Assuming these properties, it may be shewn that *linear differential equations of the form*

$$(a + bx + cx^2 \ldots)\frac{d^n u}{dx^n} + (a' + b'x + c'x^2 \ldots \frac{d^{n-1}u}{dx^{n-1}} + \&c. = X,$$

may always be reduced to the form

$$u + \phi_1(D)\,\epsilon^{\theta}u + \phi_2(D)\,\epsilon^{2\theta}u + \&c. = U \ldots\ldots (7);$$

in which $\epsilon^{\theta} = x$, U *is a function of* ϵ^{θ}, *and* $\phi_1(\mathrm{D})$, $\phi_2(\mathrm{D})$, *&c. are functions of the symbol* D *or* $\frac{\mathrm{d}}{\mathrm{d}\theta}$. This I have designated as the symbolical form of the linear differential equation: and it may be remarked, that there exists a similar form, admitting of similar treatment, in equations of finite differences.

As an example, let us take the equation

$$x^2 \frac{d^2u}{dx^2} + q^2x^2u - 6u = 0 \; \ldots\ldots\ldots \; (8),$$

which occurs in the theory of the earth's figure.

Let $x = \epsilon^\theta$. Now (Gregory's *Examples*, pp. 31, 32,)

$$x^2 \frac{d^2}{dx^2} = D(D-1).$$

Hence $$D(D-1)u + q^2\epsilon^{2\theta}u - 6u = 0,$$

$$\therefore (D+2)(D-3)u + q^2\epsilon^{2\theta}u = 0,$$

$$u + \frac{q^2}{(D+2)(D-3)}\epsilon^{2\theta}u = 0,$$

which is the symbolical form required.

From the symbolical form (7) we may at once deduce a theory of the solution of differential equations, in series extending to those cases in which the ordinary methods fail; but we shall here confine ourselves to two theorems, on which the *finite* solution of such equations chiefly depends.

PROP. 2. *When the equation* (7) *is of the form*

$$u + af(D)\epsilon^\theta u + bf(D)f(D-1)\epsilon^{2\theta}u + \&c. = 0 \ldots (9),$$

it may be resolved into a system of equations of the form

$$\left.\begin{array}{l} u - q_1 f(D)\epsilon^\theta u = 0, \\ u - q_2 f(D)\epsilon^\theta u = 0, \end{array}\right\} \ldots\ldots\ldots (10),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

q_1, q_2, *&c. being the roots of the equation*

$$q^n + aq^{n-1} + bq^{n-2} + \&c. = 0 \; \ldots\ldots \; (11).$$

To prove this, we observe, that if $f(D)\epsilon^\theta u = \rho u$, then

$$f(D)f(D-1)\epsilon^{2\theta}u = f(D)\epsilon^\theta f(D)\epsilon^\theta u = \rho^2 u, \text{ by (6)},$$

and so on; whence (9) gives

$$(1 + a\rho + b\rho^2 + \&c.)u = 0,$$

or $$(1 - q_1\rho)(1 - q_2\rho)\ldots u = 0,$$

and the theorem is manifest. The reader will easily extend it to the case in which the equation has a second member. (*Phil. Trans.* 1844, p. 245.)

This case corresponds, in the present theory, to the case of

equations with constant coefficients in the received one, but is far more general. The differential equations

$$\frac{d^n u}{dx^n} + a\frac{d^{n-1}u}{dx^{n-1}} + b\frac{d^{n-2}u}{dx^{n-2}} + \&c. = 0,$$

when reduced to the symbolical form, becomes

$$u + \frac{a}{D}\epsilon^{\theta}u + \frac{b}{D(D-1)}\epsilon^{2\theta}u + \&c. = 0,$$

which is seen to be only a particular form of (9).

3. When an equation of the symbolical form has but two terms in its first member, we can always determine whether it is integrable or not. All the integrable forms occurring in physical inquiries, with which I am acquainted, are of this class. The method of integration depends on the following theorem.

PROP. 3. *The equation* $u + \phi(D)\,\epsilon^{r\theta}u = U$, *will be converted into the form* $v + \psi(D)\,\epsilon^{r\theta}v = V$, *by the relations*

$$u = P_r\frac{\phi(D)}{\psi(D)}v, \qquad U = P_r\frac{\phi(D)}{\psi(D)}V \ldots. (12);$$

in which $P_r\dfrac{\phi(D)}{\psi(D)}$ *denotes the indefinite symbolical product*

$$\frac{\phi(D)\,\phi(D-r)\,\phi(D-2r)\ldots.}{\psi(D)\,\psi(D-r)\,\psi(D-2r)\ldots.}.$$

(*Phil. Trans.* p. 247.)

To prove this theorem, let $u = f(D)v$; and substituting in the given equation, we have

$$f(D)v + \phi(D)\,\epsilon^{r\theta}f(D)v = U,$$

$$\therefore f(D)v + \phi(D)f(D-r)\,\epsilon^{r\theta}v = U, \quad \text{by (6)},$$

or
$$v + \frac{\phi(D)f(D-r)}{f(D)}\epsilon^{r\theta}v = \{f(D)\}^{-1}U;$$

and equating coefficients,

$$\frac{\phi(D)f(D-r)}{f(D)} = \psi(D), \quad \{f(D)\}^{-1}U = V.$$

The former of these equations gives

$$f(D) = \frac{\phi(D)}{\psi(D)}f(D-r),$$

an equation of differences of the first order relative to $f(D)$, of which the solution is

$$f(D) = P_r \frac{\phi(D)}{\psi(D)};$$

whence $\quad u = P_r \dfrac{\phi(D)}{\psi(D)} v, \qquad U = P_r \dfrac{\phi(D)}{\psi(D)} V.$

We proceed to exemplify our theory in the solution of the equation of Laplace's Functions.

We have

$$\frac{d}{d\mu}(1-\mu^2)\frac{du}{d\mu} + \frac{1}{1-\mu^2}\frac{d^2u}{d\phi^2} + n(n+1)u = 0 \, . \, .(13).$$

Now ϕ only enters into this equation through the symbol of differentiation $\dfrac{d}{d\phi}$, which is commutative with respect to μ and $\dfrac{d}{d\mu}$. Let $\dfrac{d}{d\phi}\sqrt{-1} = a$, then

$$\frac{d}{d\mu}(1-\mu^2)\frac{du}{d\mu} - \frac{a^2}{1-\mu^2}u + n(n+1)u = 0 \ldots (14).$$

If we can integrate this equation regarding a as a constant, and afterwards in the most general manner interpret our result, when for a we write $\dfrac{d}{d\phi}\sqrt{-1}$, we shall evidently be in possession of the complete integral required.

Now if in (14) we write $\mu = \epsilon^\theta$, and pass to the symbolical form, we shall have an equation involving three terms in the first member, and our analysis does not in its existing state possess any *general* method of treating such equations. We must therefore endeavour, by transforming the original equation, to obviate this difficulty.

The expanded form of (14) is

$$(1-\mu^2)\frac{d^2u}{d\mu^2} - 2\mu\frac{du}{d\mu} - \frac{a^2}{1-\mu^2}u + n(n+1)u = 0 \ldots (15).$$

Assume $u = (1-\mu^2)^r v$, then

$$\frac{du}{d\mu} = (1-\mu^2)^r \frac{dv}{d\mu} - 2r\mu(1-\mu^2)^{r-1}v.$$

$$\frac{d^2u}{d\mu^2} = (1-\mu^2)^r \frac{d^2v}{d\mu^2} - 2r(1-\mu^2)^{r-1}\left(2\mu\frac{dv}{d\mu} + v\right)$$
$$+ 4r(r-1)\mu^2(1-\mu^2)^{r-2}v.$$

Substituting and effecting some reductions, we have

$$(1-\mu^2)^{r+1}\frac{d^2v}{d\mu^2}-(4r+2)\,\mu\,(1-\mu^2)^r\frac{dv}{d\mu}$$
$$+\{n(n+1)-2r-4r^2)\,(1-\mu^2)^r\,v+(4r^2-a^2)\,(1-\mu^2)^{r-1}v=0.$$

Let $4r^2-a^2=0$, then $r=\pm\frac{a}{2}$. Either sign may be taken; we choose the negative one, and suppose $r=-\frac{a}{2}$. Our equation now becomes, on dividing both sides by $(1-\mu^2)^r$,

$$(1-\mu^2)\frac{d^2v}{d\mu^2}+2(a-1)\,\mu\frac{dv}{d\mu}+\{n(n+1)-a(a-1)\}\,v=0 \quad\ldots(16).$$

In order to reduce this equation to the symbolical form, multiply by μ^2, then

$$(1-\mu^2)\,\mu^2\frac{d^2v}{d\mu^2}+2(a-1)\,\mu^3\frac{dv}{d\mu}+\{n(n+1)-a(a-1)\}\,\mu^2v=0.$$

Let $\mu=\varepsilon^\theta$, then $\mu\frac{d}{d\mu}=\frac{d}{d\theta}=D,\ \mu^2\frac{d^2}{d\mu^2}=D(D-1)$, whence

$$(1-\varepsilon^{2\theta})\,D(D-1)\,v+2(a-1)\varepsilon^{2\theta}Dv$$
$$+\{n(n+1)-a(a-1)\}\varepsilon^{2\theta}v=0 \quad\ldots\ldots(17).$$

Now $\varepsilon^{2\theta}D(D-1)\,v=(D-2)(D-3)\,\varepsilon^{2\theta}v$ by (6),

and $\varepsilon^{2\theta}Dv=(D-2)\,\varepsilon^{2\theta}v$.

Substituting these forms in (17), we have

$$D(D-1)\,v-\{(D-2)(D-3)-2(a-1)(D-2)-n(n+1)$$
$$+a(a-1)\}\,\varepsilon^{2\theta}v=0.$$

Or $D(D-1)\,v-\{D^2-(2a+3)\,D+a^2+3a-n^2-n+2\}\varepsilon^{2\theta}v=0.$

Or $\quad D(D-1)\,v-(D-a+n-1)(D-a-n-2)\,\varepsilon^{2\theta}v=0,$

on resolving the coefficient of $\varepsilon^{2\theta}v$ into its factors. Hence

$$v-\frac{(D-a+n-1)(D-a-n-2)}{D(D-1)}\,\varepsilon^{2\theta}v=0 \quad\ldots(18)$$

is the symbolical form required, and it is seen that the first member involves only two terms.

Now, let us assume

$$w-\frac{(D-a-n-1)(D-a-n-2)}{D(D-1)}\,\varepsilon^{2\theta}w=W \quad\ldots\ldots(19).$$

In these two equations v and w respectively stand for u and v of Prop. 3.

Hence $\phi(D) = \dfrac{(D-a+n-1)(D-a-n-2)}{D(D-1)}$,

$$\psi(D) = \frac{(D-a-n-1)(D-a-n-2)}{D(D-1)};$$

$$\therefore P_r \frac{\phi(D)}{\psi(D)} = P_2 \frac{D-a+n-1}{D-a-n-1}$$

$$= (D-a+n-1)(D-a+n-3)\ldots(D-a-n+1);$$

$$\therefore v = (D-a+n-1)(D-a+n-3)\ldots\ldots(D-a-n+1)\,w,$$

$$o = (D-a+n-1)(D-a+n-3)\ldots\ldots(D-a-n+1)\,W.$$

The complete value of W determined from the last equation would be

$$W = c_1 \varepsilon^{(a-n+1)\theta} + c_2 \varepsilon^{(a-n+3)\theta} + \&c.;$$

but inasmuch as the transformed equation (19) is of the same degree as the original one (18), and will therefore introduce the requisite number of arbitrary constants, it is not necessary to retain any in W, so that we have simply $W = 0$: and it is remarkable that, were we to retain all the constants which the complete value of W involves, the final value of v would not be at all affected: the unnecessary constants would disappear. We have thus to consider the system,

$$v = (D-a+n-1)(D-a+n-3)\ldots(D-a-n+1)\,w \ldots (20),$$

$$w - \frac{(D-a-n-1)(D-a-n-2)}{D(D-1)}\,\varepsilon^{2\theta} w = 0 \ldots\ldots\ldots\ldots (21).$$

The last equation may, by Prop. 2, be resolved into the system

$$\left.\begin{aligned} w + \frac{D-a-n-1}{D}\,\varepsilon^{\theta} w = 0 \\ w - \frac{D-a-n-1}{D}\,\varepsilon^{\theta} w = 0 \end{aligned}\right\} \ldots\ldots (22).$$

From the former of these equations we have

$$Dw + (D-a-n-1)\,\varepsilon^{\theta} w = 0\,;$$

or
$$Dw + \varepsilon^{\theta}(D-a-n)\,w = 0.$$

$$\therefore \frac{dw}{d\theta} - \frac{(a+n)\,\varepsilon^{\theta}}{1+\varepsilon^{\theta}}\,w = 0,$$

$$w = (1+\varepsilon^{\theta})^{n+a}\,\psi(\phi),$$

$\psi(\phi)$ denoting an arbitrary function of ϕ. In like manner, from the second equation of the system (22), we have

$$w = (1-\varepsilon^{\theta})^{n+a}\,\chi(\phi).$$

Hence the complete value of w is

$$w = (1+\varepsilon^{\theta})^{n+a}\psi(\phi) + (1-\varepsilon^{\theta})^{n+a}\chi(\phi)$$
$$= (1+\mu)^{n+a}\psi(\phi) + (1-\mu)^{n+a}\chi(\phi) \;\ldots\ldots(23).$$

To simplify the expression for v we observe that, in general,

$$f(D)\,U = f(D)\,\varepsilon^{r\theta}\varepsilon^{-r\theta}U$$
$$= \varepsilon^{r\theta}f(D+r)\,\varepsilon^{-r\theta}U \text{ by (6).}$$

Now inverting the order of the factors in the right-hand member of (20),

$$v = (D-a-n+1)(D-a-n+3)\ldots(D-a+n-1)\,w$$
$$= \varepsilon^{(a+n-1)\theta}D(D+2)\ldots(D+2n-2)\,\varepsilon^{-(a+n-1)\theta}w.$$

But $D+2 = \varepsilon^{-2\theta}D\varepsilon^{2\theta}$, $D+4 = \epsilon^{-4\theta}D\varepsilon^{4\theta}$, and so on. Substituting, we have

$$v = \varepsilon^{(a+n-1)\theta}D\varepsilon^{-2\theta}D\varepsilon^{-2\theta}\ldots D\varepsilon^{(2n-2)\theta}\varepsilon^{-(a+n-1)\theta}w.$$

Making $\varepsilon^{\theta}=\mu$, $D=\mu\dfrac{d}{d\mu}$, the above equation assumes the form

$$v = \mu^{n+a}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^{n}\mu^{n-a}w \;\ldots\ldots\ldots\ldots(24).$$

Now $u=(1-\mu^2)^{-\frac{a}{2}}v$; hence, writing in full the values of v and w, we have

$$u = (1-\mu^2)^{-\frac{a}{2}}\mu^{n+a}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^{n}\mu^{n-a}\left\{(1+\mu)^{n+a}\psi(\phi)+(1-\mu)^{n+a}\chi(\phi)\right\}$$
$$\ldots\ldots\ldots(25).$$

It remains to interpret this remarkable expression.

As $\psi(\phi)$, $\chi(\phi)$, are perfectly arbitrary, it is obvious that we may, in place of them, write $\psi(\epsilon^{\phi\sqrt{-1}})$, $\chi(\epsilon^{\phi\sqrt{-1}})$. The interpretation of our formula does not require this transformation, but the result is thereby simplified. We are thus at liberty to express the integral in the following form:

$$u = \mu^{n}\left(\frac{\mu}{\sqrt{(1-\mu^2)}}\right)^{a}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^{n}\left\{(\mu+\mu^2)^{n}\left(\frac{1+\mu}{\mu}\right)^{a}\psi(\epsilon^{\phi\sqrt{+1}})\right.$$
$$\left.+(\mu-\mu^2)^{n}\left(\frac{1-\mu}{\mu}\right)^{a}\chi(\epsilon^{\phi\sqrt{-1}})\right\}\;\ldots\ldots(26),$$

or by two equations thus,

$$u = \left\{\frac{\mu}{\sqrt{(1-\mu^2)}}\right\}^{a}F(\mu,\,\epsilon^{\phi\sqrt{-1}}),$$

where $F(\mu,\ \epsilon^{\phi\sqrt{-1}}) = \mu^n \left(\dfrac{d}{d\mu}\dfrac{1}{\mu}\right)^n \left\{(\mu+\mu^2)^n \left(\dfrac{1+\mu}{\mu}\right)^a \psi(\epsilon^{\phi\sqrt{-1}})\right.$

$$\left. + (\mu-\mu^2)^n \left(\frac{1-\mu}{\mu}\right)^a \chi(\epsilon^{\phi\sqrt{-1}})\right\} \dots\dots (27).$$

Now t being any quantity independent of ϕ, we have

$$t^a f(\epsilon^{\phi\sqrt{-1}}) = t^{\frac{d}{d\phi}\sqrt{-1}} f(\epsilon^{\phi\sqrt{-1}})$$

$$= \epsilon^{\sqrt{-1}\log t\,\frac{d}{d\phi}} f(\epsilon^{\phi\sqrt{-1}})$$

$$= f(\epsilon^{(\phi+\sqrt{-1}\log t)\sqrt{-1}}), \text{ by Taylor's theorem,}$$

$$= f\left(\frac{\epsilon^{\phi\sqrt{-1}}}{t}\right).$$

Hence $$\left(\frac{1+\mu}{\mu}\right)^a \psi(\epsilon^{\phi\sqrt{-1}}) = \psi\left(\frac{\mu\,\epsilon^{\phi\sqrt{-1}}}{1+\mu}\right),$$

and so on; whence finally

$$u = F\left(\mu,\ \frac{\sqrt{(1-\mu^2)}}{\mu}\,\epsilon^{\phi\sqrt{-1}}\right),$$

where $F(\mu,\ \epsilon^{\phi\sqrt{-1}})$

$$= \mu^n \left(\frac{d}{d\mu}\frac{1}{\mu}\right)^n \left\{(\mu+\mu^2)^n \psi\left(\frac{\mu\,\epsilon^{\phi\sqrt{-1}}}{1+\mu}\right) + (\mu-\mu^2)^n \chi\left(\frac{\mu\,\epsilon^{\phi\sqrt{-1}}}{1-\mu}\right)\right\} \dots (28),$$

which is the complete integral required. The symbol $\left(\dfrac{d}{d\mu}\dfrac{1}{\mu}\right)^n$, of course, implies the performance of n successive operations, each of which is effected by dividing the subject by μ, and then taking the differential coefficient with respect to that variable.

Discussion of the Integral.

If we represent the two particular integrals in the above general solution by U and V respectively, then in U, supposing $\psi(\epsilon^{\phi\sqrt{-1}}) = \epsilon^{r\phi\sqrt{-1}}$, we shall have

$$U = \left\{\frac{\sqrt{(1-\mu^2)}\,\epsilon^{\phi\sqrt{-1}}}{\mu}\right\}^r \mu^n \left(\frac{d}{d\mu}\frac{1}{\mu}\right)^n (\mu+\mu^2)^n \left(\frac{\mu}{1+\mu}\right)^r$$

$$= (1-\mu^2)^{\frac{r}{2}} \mu^{n-r} \left(\frac{d}{d\mu}\frac{1}{\mu}\right)^n \mu^{n+r} (1+\mu)^{n-r} \epsilon^{r\phi\sqrt{-1}}$$

$$= f(\mu)\,\epsilon^{r\phi\sqrt{-1}}, \text{ for abbreviation.} \dots\dots\dots\dots (29)$$

Again, in U let $\psi(\epsilon^{\phi\sqrt{-1}}) = \epsilon^{-r\phi\sqrt{-1}}$, we have

$$U = \left\{\frac{\sqrt{(1-\mu^2)}\,\epsilon^{\phi\sqrt{-1}}}{\mu}\right\}^{-r} \mu^n \left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n (\mu+\mu^2)^n \left(\frac{\mu}{1+\mu}\right)^{-r}$$

$$= (1-\mu^2)^{-\frac{r}{2}} \mu^{n+r} \left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n \mu^{n-r} (1+\mu)^{n+r} \epsilon^{-r\phi\sqrt{-1}}$$

$$= f_1(\mu)\,\epsilon^{-r\phi\sqrt{-1}}, \text{ for abbreviation. } \ldots\ldots\ldots (30).$$

The functions $f_1(\mu)$, $f(\mu)$, have a very singular relation, viz.

$$f_1(\mu) = (-)^n f(-\mu) \ldots\ldots\ldots\ldots (31),$$

which may be proved thus. The two first terms of the expansion of $f(\mu)$, in ascending powers of μ, are

$$(r+n-1)(r+n-3)\ldots(r-n+1) - (r+n)(r+n-2)\ldots(r-n)\,\mu \ldots (32);$$

and the two first terms in the expansion of $f_1(\mu)$ are

$$(-)^n \{(r+n-1)(r+n-3)\ldots(r-n+1) + (r+n)(r+n-2)\ldots(r-n)\,\mu\} \ldots . (33).$$

But $f(\mu)\,\epsilon^{r\phi\sqrt{-1}}$, and $f_1(\mu)\,\epsilon^{-r\phi\sqrt{-1}}$ are both solutions of the differential equation given. If we substitute them in that equation in the forms

$$u = \Sigma u_m \mu^m \epsilon^{r\phi\sqrt{-1}}, \quad u = \Sigma u_m \mu^m \epsilon^{-r\phi\sqrt{-1}},$$

we shall find that the successive values of u_m are in each case connected by the relation

$$u_m - \frac{2(m-2)^2 - n(n+1) + 2r^2}{m(m-1)} u_{m-2} + \frac{(m+n-3)(m-n-4)}{m(m-1)} u_{m-4} = 0 \ldots . (34),$$

the values u_0 and u_1 being arbitrary. From the relation just given it appears, that all the even coefficients, u_2, u_4, &c., will be formed from u_0, and all the odd ones, u_3, u_5, &c., from u_1. Hence, on comparing (32) and (33), we see that the even terms in the expansion of $f_1(\mu)$ will differ only from those in the expansion of $f(\mu)$ by the sign $(-)^n$, and that the odd terms will only differ by the sign $(-)^{n+1}$; and as the odd terms change sign with μ, the relation between $f_1(\mu)$ and $f(\mu)$ will be such as we have assigned in (31).

From the above it appears that the assumption

$$\psi(\epsilon^{\phi\sqrt{-1}}) = a\,\epsilon^{r\phi\sqrt{-1}} + b\,\epsilon^{-r\phi\sqrt{-1}},$$

will give

$$U = af(\mu)\,\epsilon^{r\phi\sqrt{-1}} + (-)^n bf(-\mu)\,\epsilon^{-r\phi\sqrt{-1}} \ldots . (35);$$

and, by an exactly similar process of reasoning and comparison, we shall find that if, in the second integral V, we assume

$$\chi(\epsilon^{\phi\sqrt{-1}}) = a'\,\epsilon^{r\phi\sqrt{-1}} + b'\,\epsilon^{-r\phi\sqrt{-1}},$$

then
$$V = a'f(-\mu)\,\epsilon^{r\phi\sqrt{-1}} + (-)^n\,b'f(\mu)\,\epsilon^{-r\phi\sqrt{-1}}\ldots.(36).$$

Adding these values,

$$\begin{aligned} u &= \{af(\mu) + a'f(-\mu)\}\,\epsilon^{r\phi\sqrt{-1}} + (-)^n\{b'f(\mu) + bf(-\mu)\}\,\epsilon^{-r\phi\sqrt{-1}} \\ &= \left.\begin{array}{l} [\{a + (-)^n b'\}f(\mu) + \{a' + (-)^n b\}f(-\mu)]\cos r\phi \\ +[\{a - (-)^n b'\}f(\mu) + \{a' - (-)^n b\}f(-\mu)]\sqrt{-1}\,\sin r\phi \end{array}\right\}\ldots(37). \end{aligned}$$

If we assume

$$a - (-)^n b' = 0, \qquad a' - (-)^n b = 0,$$

we have
$$u = 2\{af(\mu) + a'f(-\mu)\}\cos r\phi\ldots.(38):$$

and as this is true for all values of r, it is clear that, had we assumed $\psi(\epsilon^{\phi\sqrt{-1}}) = \Sigma(a\epsilon^{r\phi\sqrt{-1}} + b\epsilon^{-r\phi\sqrt{-1}})$, &c., we should have had, in place of (38),

$$u = \Sigma\{2af(\mu) + 2a'f(-\mu)\}\cos r\phi;$$

or, putting $2a = c$, $2a' = c'$,

$$u = \Sigma\{cf(\mu) + c'f(-\mu)\}\cos r\phi\ldots\ldots(39),$$

the values of c and c' differing for different values of r.

This is the most general form, *of the kind*, which the integral can assume. It is remarkable that the coefficient of $\cos r\phi$ will be always a *finite algebraic* function of μ, whether r be integral or not, its expression being

$$f(\mu) = (1-\mu^2)^{\frac{r}{2}}\,\mu^{n-r}\left(\frac{d}{d\mu}\,\frac{1}{\mu}\right)^n \mu^{n+r}\,(1+\mu)^{n-r}\ldots.(40).$$

It is also to be remarked that, by attributing other forms to ψ and χ, and especially *logarithmic* forms, we can obtain an infinite variety of finite solutions of a character altogether different from the above.

Forms of Laplace's Coefficients.

In the case we have now to consider, u, or as it is commonly written, P_n, is the coefficient of t^n in the development of the function

$$[1 - 2\{\mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)}\cos\phi\}\,t + t^2]^{-\frac{1}{2}}\ldots(41),$$

ϕ standing for $\phi - \phi'$ in the ordinary treatises.

Here it is evident, that the values of r in $\cos r\phi$ are all integral, and that its limits will be 0 and n. Hence (39) gives

$$P_n = \Sigma_{r=0}^{r=n} \{cf(\mu) + c'f(-\mu)\} \cos r\phi.$$

To determine the coefficients, we must examine the first term of the expanded coefficient of $t^n \cos r\phi$ in the expansion of (41). This is easily found to be either

$$2\,\frac{1.3 \ldots n+r-1}{2.4 \ldots n+r} \times \frac{1.3 \ldots n-r-1}{2.4 \ldots n-r},$$

$$\text{or } 2\,\frac{1.3 \ldots n+r}{2.4 \ldots n+r-1} \times \frac{3.5 \ldots n-r}{2.4 \ldots n-r-1}\,\mu\mu' \ldots . (42),$$

according as $n-r$ is even or odd.

Again, the first term of the expansion of $f(\mu)$ is, in this case,

$$(r+n-1)(r+n-3) \ldots (r-n+1),$$

$$\text{or } -(r+n)(r+n-2) \ldots (r-n)\,\mu \ldots \ldots (43),$$

according as $n-r$ is even or odd, the remaining term vanishing. If $n-r$ be even, it is clear from (34) that, as the first term involving an odd index in $f(\mu)$ vanishes, all such terms will vanish. Here then $f(-\mu) = f(\mu)$. In the case of $n-r$ being odd, it is evident that all the terms in the expansion of $f(\mu)$ will have odd indices, and $f(-\mu) = -f(\mu)$. In either case we have

$$P_n = \Sigma_{r=0}^{r=n}\, af(\mu) \cos r\phi,$$

a being a constant, *i.e.* a quantity independent of μ and ϕ. Now, from the form of (41), it is evident that μ and μ' are symmetrically involved in P_n. Hence $a = a_r f(\mu')$, and

$$P_n = \Sigma_{r=0}^{r=n}\, a_r f(\mu) f(\mu') \cos r\phi.$$

If $n-r$ be even, the first term of the expansion of $a_r f(\mu) f(\mu')$ will be

$$a_r \{(r+n-1)(r+n-3) \ldots (r-n+1)\}^2;$$

which, equated with the corresponding term in (42), gives

$$a_r = \frac{2}{1.2 \ldots n+r \;\; 1.2 \ldots n-r},$$

except when $r = 0$; in which case

$$a_0 = \frac{1}{(1.2 \ldots n)^2}.$$

If $n - r$ be odd, the first term of the expansion of $a_r f(\mu) f(\mu')$ will be $a_r \{(r+n)(r+n-2)\ldots(r-n)\}^2$; which, equated with the corresponding term in (42), leads to the same result.

Hence, finally, we have

$$P_n = A_0 + 2(A_1 \cos\phi + A_2 \cos 2\phi \ldots + A_n \cos n\phi) \ldots (44),$$

where any coefficient A_r is of the form

$$A_r = \frac{f(\mu) f(\mu')}{1.2 \ldots n+r \;\; 1.2 \ldots n-r},$$

and in general

$$f(\mu) = (1 - \mu^2)^{\frac{r}{2}} \mu^{n-r} \left(\frac{d}{d\mu} \frac{1}{\mu}\right)^n \mu^{n+r} (1 + \mu)^{n-r}.$$

I have entered with more particularity into the details of the above solution, than to some might have appeared necessary; but it was my object in this paper, not only to integrate the Equation of Laplace, but also to illustrate, and in so doing, if it might be, to recommend a method in Analysis.

Lincoln, July 28, 1845.

ON SOME ANALYTICAL FORMULÆ, AND THEIR APPLICATION TO THE THEORY OF SPHERICAL CO-ORDINATES.

By ARTHUR CAYLEY, M.A., Fellow of Trinity College, Cambridge.

Section 1.

THE formulæ in question are only very particular cases of some relating to the theory of the transformation of functions of the second order, which will be given in a following paper. But the case of three variables, here as elsewhere, admits of a symmetrical notation so much simpler than in any other case (on the principle that with three quantities a, b, c, functions of b, c; of c, a; and a, b, may symmetrically be denoted by A, B, C, which is not possible with a greater number of variables) that it will be convenient to employ here a notation entirely different from that made use of in the general case, and by means of which the results will be exhibited in a more compact form. There is no difficulty in verifying by actual multiplication, any of the equations here obtained.

It will be expedient to employ the abbreviation of making a single letter stand for a system of quantities. Thus for instance, if $\aleph = \theta, \phi, \psi$, this merely means that $\Phi(\aleph)$ is to stand for $\Phi(\theta, \phi, \psi)$, $k\aleph$ for $k\theta, k\phi, k\psi$, &c.

Suppose then $$\omega = \xi, \eta, \zeta, \quad \ldots \ldots \quad (1),$$
$$\omega' = \xi', \eta', \zeta',$$
$$\vdots$$
$$Q = A, B, C, F, G, H. \quad \ldots \quad (2),$$
$$W(\omega, \omega', Q) = A\xi\xi' + B\eta\eta' + C\zeta\zeta'$$
$$+ F(\eta\zeta' + \eta'\zeta) + G(\zeta\xi' + \zeta'\xi) + H(\xi\eta' + \xi'\eta) \ldots (3).$$

The function W satisfies a remarkable equation, as follows.

Write $$\mathfrak{A} = BC - F^2, \quad \ldots \ldots \quad (4),$$
$$\mathfrak{B} = CA - G^2,$$
$$\mathfrak{C} = AB - H^2.$$
$$\mathfrak{F} = GH - AF,$$
$$\mathfrak{G} = HF - BG,$$
$$\mathfrak{H} = FG - CH.$$
$$\mathfrak{Q} = \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \quad \ldots \quad (5).$$
$$\overline{\omega\omega'} = \eta\zeta' - \eta'\zeta,\ \zeta\xi' - \zeta'\xi,\ \xi\eta' - \xi'\eta \quad \ldots \quad (6).$$

We have
$$W(\omega_1, \omega_2, Q)\ W(\omega_3, \omega_4, Q) - W(\omega_1, \omega_3, Q)\ W(\omega_2, \omega_4, Q)$$
$$= W(\overline{\omega_1\omega_4}, \overline{\omega_2\omega_3}, \mathfrak{Q}) \ldots (7),$$

of which we may notice also the particular cases
$$W(\omega_1\omega_2, Q)\ W(\omega_3\omega_3, Q) - W(\omega_1\omega_3, Q)\ W(\omega_2, \omega_3, Q)$$
$$= W(\overline{\omega_1\omega_3}, \overline{\omega_2\omega_3}, \mathfrak{Q}) \ldots (8),$$
$$W(\omega_1, \omega_1, Q)\ W(\omega_2, \omega_2, Q) - \{W(\omega_1\omega_2, Q)\}^2$$
$$= W(\overline{\omega_1\omega_2}, \overline{\omega_1\omega_2}, \mathfrak{Q}) \ldots (9).$$

To which we may join the following formulæ, for the transformation of the function W.

Suppose
$$\omega_1 = \mathrm{a}x_1 + \mathrm{a}'y_1 + \mathrm{a}''z_1,\ \mathrm{b}x_1 + \mathrm{b}'y_1 + \mathrm{b}''z_1,\ \mathrm{c}x_1 + \mathrm{c}'y_1 + \mathrm{c}''z_1 \ldots (10).$$
$$\omega_2 = \mathrm{a}x_2 + \mathrm{a}'y_2 + \mathrm{a}''z_2,\ \mathrm{b}x_2 + \mathrm{b}'y_2 + \mathrm{b}''z_2,\ \mathrm{c}x_2 + \mathrm{c}'y_2 + \mathrm{c}''z_2.$$
$$\vdots$$

Then, writing $$g = \mathrm{a}, \mathrm{b}, \mathrm{c} \quad \ldots \ldots \quad (11)$$
$$g' = \mathrm{a}', \mathrm{b}', \mathrm{c}',$$
$$g'' = \mathrm{a}'', \mathrm{b}'', \mathrm{c}''.$$
$$p_1 = x_1, y_1, z_1 \quad \ldots \ldots \quad (12).$$
$$p_2 = x_2, y_2, z_2,$$
$$\vdots$$
$$\Theta = W(g, g, Q),\ W(g', g', Q),\ W(g'', g'', Q),$$
$$W(g', g'', Q),\ W(g'', g, Q),\ W(g, g', Q) \ldots (13).$$

We have $$W(\omega_1, \omega_2, Q) = W(p_1, p_2, \Theta) \quad . \quad . \quad . \quad (14).$$
Similarly, writing

$$\Psi = W(\overline{\underline{g'g''}}, \overline{\underline{g'g''}}, \mathfrak{Q}),\ W(\overline{\underline{g''g}}, \overline{\underline{g''g}}, \mathfrak{Q}),\ W(\overline{\underline{gg'}}, \overline{\underline{gg'}}, \mathfrak{Q}) \,.\,.\, (15).$$

$$W(\overline{\underline{gg'}}, \overline{\underline{g''g}}, \mathfrak{Q}),\quad W(\overline{\underline{g'g''}}, \overline{\underline{gg'}}, \mathfrak{Q}),\quad W(\overline{\underline{g''g}}, \overline{\underline{g'g''}}, \mathfrak{Q}).$$

we have $$W(\overline{\underline{\omega_1\omega_4}}, \overline{\underline{\omega_2\omega_3}}, \mathfrak{Q}) = W(\overline{\underline{p_1p_4}}, \overline{\underline{p_2p_3}}, \Psi) \,.\,.\,.\,.\, (16),$$

in which equations $\mathfrak{Q}$ may obviously be changed into Q.

Section 2.—Geometrical Applications.

Consider any three axes Ax, Ay, Az, and let λ, μ, ν be the cosines of the inclinations of these lines to each other.

Let Λ, M, N be the inclinations of the co-ordinate planes to each other; l, m, n, the inclination of the axes to the co-ordinate planes. Suppose, besides,

$$\begin{aligned}
\mathfrak{a} &= 1 - \lambda^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17).\\
\mathfrak{b} &= 1 - \mu^2,\\
\mathfrak{c} &= 1 - \nu^2,\\
\mathfrak{f} &= \mu\nu - \lambda,\\
\mathfrak{g} &= \nu\lambda - \mu,\\
\mathfrak{h} &= \lambda\mu - \nu.
\end{aligned}$$

$$k = 1 - \lambda^2 - \mu^2 - \nu^2 + 2\lambda\mu\nu \quad . \quad . \quad . \quad (18).$$

We have the following systems of equations:

$$\begin{aligned}
&\sqrt{(\mathfrak{bc})} \cos \Lambda = -\mathfrak{f}, &&\sqrt{(\mathfrak{bc})} \sin \Lambda = \sqrt{(k)}, &&\sqrt{(\mathfrak{a})} \sin l = \sqrt{(k)} \,.\,.\, (19).\\
&\sqrt{(\mathfrak{ca})} \cos \mathrm{M} = -\mathfrak{g}, &&\sqrt{(\mathfrak{ca})} \sin \mathrm{M} = \sqrt{(k)}, &&\sqrt{(\mathfrak{b})} \sin m = \sqrt{(k)}\\
&\sqrt{(\mathfrak{ab})} \cos \mathrm{N} = -\mathfrak{h}, &&\sqrt{(\mathfrak{ab})} \sin \mathrm{N} = \sqrt{(k)}, &&\sqrt{(\mathfrak{c})} \sin n = \sqrt{(k)}.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{a} + \nu\mathfrak{h} + \mu\mathfrak{g} &= k, \quad . \quad . \quad . \quad . \quad . \quad (20).\\
\nu\mathfrak{a} + \mathfrak{h} + \lambda\mathfrak{g} &= 0,\\
\mu\mathfrak{a} + \lambda\mathfrak{h} + \mathfrak{g} &= 0,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{h} + \nu\mathfrak{b} + \mu\mathfrak{f} &= 0, \quad . \quad . \quad . \quad . \quad . \quad (21).\\
\nu\mathfrak{h} + \mathfrak{b} + \lambda\mathfrak{f} &= k,\\
\mu\mathfrak{h} + \lambda\mathfrak{b} + \mathfrak{f} &= 0.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{g} + \nu\mathfrak{f} + \mu\mathfrak{c} &= 0, \quad . \quad . \quad . \quad . \quad . \quad (22).\\
\nu\mathfrak{g} + \mathfrak{f} + \lambda\mathfrak{c} &= 0,\\
\mu\mathfrak{g} + \lambda\mathfrak{f} + \mathfrak{c} &= k.
\end{aligned}$$

$$\mathfrak{bc} - \mathfrak{f}^2 = ka \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$
$$\mathfrak{ca} - \mathfrak{g}^2 = kb,$$
$$\mathfrak{ab} - \mathfrak{h}^2 = kc,$$
$$\mathfrak{gh} - \mathfrak{af} = kf,$$
$$\mathfrak{hf} - \mathfrak{bg} = kg,$$
$$\mathfrak{fg} - \mathfrak{ch} = kh,$$
$$\mathfrak{abc} - \mathfrak{af}^2 - \mathfrak{bg}^2 - \mathfrak{ch}^2 + 2\mathfrak{fgh} = k^2 \quad . \quad . \quad . \quad (24).$$

Imagine now a line AO, and let α, β, γ be the cosines of its inclinations to the three axes. Suppose also, θ, ϕ, χ being its inclinations to the co-ordinate planes, we write

$$a = \frac{\sin\theta}{\sqrt{(\mathfrak{a})}}, \quad b = \frac{\sin\phi}{\sqrt{(\mathfrak{b})}}, \quad c = \frac{\sin\chi}{\sqrt{(\mathfrak{c})}} \quad . \quad . \quad . \quad (25).$$

If we consider a point P on the line AO, at a distance unity from the origin, we see immediately, by considering the projections in the directions perpendicular to the co-ordinate planes, that the co-ordinates of this point are a, b, c. By projecting on the three axes and on the line AO, we then obtain the equations

$$\alpha = a + \nu b + \mu c \quad . \quad . \quad . \quad . \quad . \quad . \quad (26).$$
$$\beta = \nu a + b + \lambda c,$$
$$\gamma = \mu a + \lambda b + c,$$
$$1 = \alpha a + \beta b + \gamma c \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

From which we obtain

$$ka = \mathfrak{a}\alpha + \mathfrak{h}\beta + \mathfrak{g}\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$
$$kb = \mathfrak{h}\alpha + \mathfrak{b}\beta + \mathfrak{f}\gamma,$$
$$kc = \mathfrak{g}\alpha + \mathfrak{f}\beta + \mathfrak{c}\gamma,$$
$$1 = \alpha a + \beta b + \gamma c \quad . \quad . \quad . \quad . \quad . \quad . \quad (29).$$

And hence

$$1 = a^2 + b^2 + c^2 + 2\lambda bc + 2\mu ac + 2\nu ab \quad . \quad . \quad (30).$$
$$k = \mathfrak{a}\alpha^2 + \mathfrak{b}\beta^2 + \mathfrak{c}\gamma^2 + 2\mathfrak{f}\beta\gamma + 2\mathfrak{g}\alpha\gamma + 2\mathfrak{h}\alpha\beta \quad . \quad . \quad (31).$$

Or writing $\qquad a, b, c = t \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$

$$\alpha, \beta, \gamma = \tau \quad . \quad . \quad . \quad . \quad . \quad . \quad (33).$$
$$1, 1, 1, \lambda, \mu, \nu = \mathrm{q} \quad . \quad . \quad . \quad . \quad . \quad (34).$$
$$\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{f}, \mathfrak{g}, \mathfrak{h} = \mathfrak{q} \quad . \quad . \quad . \quad . \quad . \quad (35).$$

we have the equations $\quad 1 = W(t, t, \mathrm{q}) \quad . \quad . \quad . \quad . \quad . \quad (36).$

$$k = W(\tau, \tau, \mathfrak{q}) \quad . \quad . \quad . \quad . \quad . \quad (37).$$

Let AO' be any other line, and δ its inclination to AO: α', β', γ', a', b', c', the quantities corresponding to α, β, γ, a, b, c, and similarly t', τ' to t, τ. We have of course

$$1 = W(t', t', \mathrm{q}) \quad . \quad . \quad . \quad . \quad . \quad (38),$$

$$k = W(\tau', \tau', \mathfrak{q}) \quad . \quad . \quad . \quad . \quad . \quad (39).$$

We have besides, by projecting on the line AO', the equation

$$\cos\delta = \alpha' a + \beta' b + \gamma' c \quad . \quad . \quad . \quad (40),$$

or the analogous one

$$\cos\delta = \alpha' a + \beta' b + \gamma' c \quad . \quad . \quad . \quad (41).$$

From either of which we deduce

$$\cos\delta = aa' + bb' + cc' + \lambda.(bc' + b'c) + \mu.(ca' + c'a) + \nu.(ab' + a'b) \ldots (42),$$

$$k\cos\delta = \mathfrak{a}\alpha\alpha' + \mathfrak{b}\beta\beta' + \mathfrak{c}\gamma\gamma' + \mathfrak{f}(\beta\gamma' + \beta'\gamma) + \mathfrak{g}(\gamma\alpha' + \gamma'\alpha) + \mathfrak{h}(\alpha\beta + \alpha'\beta) \ldots (43);$$

which may otherwise be written

$$\cos\delta = W(t, t', \mathrm{q}) \quad . \quad . \quad . \quad . \quad . \quad (44),$$

$$k\cos\delta = W(\tau, \tau', \mathfrak{q}) \quad . \quad . \quad . \quad . \quad . \quad (45).$$

Or again, observing the equations which connect the quantities t, τ,

$$\cos\delta = \frac{W(t, t', \mathrm{q})}{\sqrt{\{W(t, t, \mathrm{q}).W(t', t', \mathrm{q})\}}} \quad . \quad . \quad (46),$$

$$\cos\delta = \frac{W(\tau, \tau', \mathfrak{q})}{\sqrt{\{W(\tau, \tau, \mathfrak{q}).W\tau', \tau', \mathfrak{q}\}}} \quad . \quad . \quad (47),$$

forms which, though more complicated, have certain advantages; for instance, we derive immediately from them the new equations

$$\sin\delta = \frac{\sqrt{\{W(\overline{tt'}, \overline{tt'}, \mathfrak{q})\}}}{\sqrt{\{W(t, t, \mathrm{q})\, W(t', t', \mathrm{q})\}}} \quad . \quad . \quad (48),$$

$$\sin\delta = \frac{\sqrt{\{kW(\overline{\tau\tau'}, \overline{\tau\tau'}, \mathrm{q})\}}}{\sqrt{\{W(\tau, \tau, \mathfrak{q})\, W(\tau', \tau', \mathfrak{q})\}}} \quad . \quad . \quad (49).$$

Written more simply thus

$$\sin\delta = W(\overline{tt'}, \overline{tt'}, \mathfrak{q}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

$$\sqrt{k}\sin\delta = \sqrt{\{W(\overline{\tau\tau'}, \overline{\tau\tau'}, \mathrm{q})\}} \quad . \quad . \quad . \quad . \quad (51),$$

to which we may join

$$\cot\delta = \frac{W(t, t', \mathrm{q})}{\sqrt{\{W(\overline{tt'}, \overline{tt'}, \mathfrak{q})\}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (52),$$

$$\sqrt{k} \cot \delta = \frac{W(\tau, \tau', \mathfrak{q})}{\sqrt{\{W(\overline{\underline{\tau\tau'}}, \overline{\underline{\tau\tau'}}, \mathfrak{q})\}}} \quad . \quad . \quad . \quad . \quad (53).$$

Section 3.—On Spherical Coordinates.

Consider the points X, Y, Z, on the surface of a sphere, as the intersections of the three axes of the preceding section, with a sphere having its centre in the origin. It is evident that λ, μ, ν are the cosines of the sides of the spherical triangle XYZ, $\Lambda, \mathrm{M}, \mathrm{N}$ are its sides, l, m, n are the perpendiculars from the angles upon the opposite sides. Let P be the point where the line AO intersects the sphere: the position of the point P may be determined by means of the ratios $\xi : \eta : \zeta$, supposing ξ, η, ζ denote quantities proportional to the α, β, γ of the preceding section, i.e.

$$\xi : \eta : \zeta = \cos PX : \cos PY : \cos PZ \quad . \quad . \quad (54).$$

Or again, by means of the ratios $x : y : z$, supposing x, y, z denote quantities proportional to the a, b, c of the preceding section, i.e.

$$x : y : z = \frac{\sin Px}{\sin X} : \frac{\sin Py}{\sin Y} : \frac{\sin Pz}{\sin Z} \quad . \quad . \quad (55),$$

(Px, Py, Pz are the perpendiculars from P on the sides of the spherical triangle XYZ).

Which equations may be otherwise written,

$$\frac{x \sin X}{y \sin Y} = \frac{\sin PZY}{\sin PZX} \quad . \quad . \quad . \quad . \quad . \quad (56).$$

$$\frac{y \sin Y}{z \sin Z} = \frac{\sin PXZ}{\sin PXY},$$

$$\frac{z \sin Z}{x \sin X} = \frac{\sin PYX}{\sin PYZ}.$$

The ratios $\xi : \eta : \zeta$, or $x : y : z$, are termed the spherical coordinate ratios of the point P. The two together may be termed conjoint systems: the first may be termed the cosine system, and the second the sine system. The coordinates of the two systems are evidently connected by

$$\xi : \eta : \zeta = x + \nu y + \mu z : \nu x + y + \lambda z : \mu x + \lambda y + z \ldots (57),$$

or $\quad x : y : z = \mathfrak{a}\xi + \mathfrak{h}\eta + \mathfrak{g}\zeta : \mathfrak{h}\xi + \mathfrak{b}\eta + \mathfrak{f}\zeta : \mathfrak{g}\xi + \mathfrak{f}\eta + \mathfrak{c}\zeta \ldots (58).$

The systems may conveniently be represented by the single letters

$$\omega = \xi, \eta, \zeta \quad . \quad . \quad . \quad . \quad . \quad . \quad (59),$$

$$p = x, y, z \quad . \quad . \quad . \quad . \quad . \quad . \quad (60).$$

Fundamental formula of spherical coordinates; distance of two points.

Let P, P' be the points, δ their distance, ω, p the conjoint coordinate systems of the first point, ω', p' of the second; we have obviously

$$\cos\delta = \frac{W(p, p', q)}{\sqrt{\{W(p, p, q)\, W(p', p', q)\}}} \quad \ldots\ldots (61),$$

$$\sin\delta = \frac{\sqrt{\{W(\overline{pp'}, \overline{pp'}, \mathfrak{q})\}}}{\sqrt{\{W(p, p, q)\, W(p', p', q)\}}},$$

$$\cot\delta = \frac{W(p, p', q)}{\sqrt{\{W(\overline{pp'}, \overline{pp'}, \mathfrak{q})\}}};$$

or
$$\cos\delta = \frac{W(\omega, \omega', \mathfrak{q})}{\sqrt{\{W(\omega, \omega, \mathfrak{q})\, W(\omega', \omega', \mathfrak{q})\}}} \quad \ldots\ldots (62).$$

$$\frac{1}{\sqrt{k}}\sin\delta = \frac{\sqrt{\{W(\overline{\omega\omega'}, \overline{\omega\omega'}, q)\}}}{\sqrt{\{W(\omega, \omega, \mathfrak{q})\, W(\omega', \omega', \mathfrak{q})\}}},$$

$$\sqrt{k}\cot\delta = \frac{W(\omega, \omega')\, \mathfrak{q})}{\sqrt{\{W(\overline{\omega\omega'}, \overline{\omega\omega'}, q)\}}}.$$

Equation of a great Circle.

Let the conjoint coordinate systems of the pole be

$$e = a, b, c \quad \ldots\ldots\ldots\ldots\ldots (63),$$

$$\epsilon = \alpha, \beta, \gamma \quad \ldots\ldots\ldots\ldots\ldots (64).$$

Then, expressing that the distance of any point P in the locus from the pole is equal to $90°$, we have immediately the equations

$$W(p, e, q) = 0 \quad \ldots\ldots\ldots\ldots (65),$$

$$W(\omega, \epsilon, \mathfrak{q}) = 0 \quad \ldots\ldots\ldots\ldots (66),$$

which may otherwise be written in the forms

$$a\xi + b\eta + c\zeta = 0 \quad \ldots\ldots\ldots\ldots (67),$$

$$\alpha x + \beta y + \gamma\zeta = 0 \quad \ldots\ldots\ldots\ldots (68),$$

or the equation of a great circle is linear in either coordinate system. Conversely, any linear equation belongs to a great circle.

Suppose the equation given in the form

$$A\xi + B\eta + C\zeta = 0 \quad \ldots\ldots\ldots\ldots (69);$$

or by an equation between cosine coordinate ratios:—The sine system for the pole is given by

$$e = A, B, C \quad \ldots\ldots\ldots\ldots\ldots (70),$$

and the cosine system by

$$\epsilon = A + \nu B + \mu C, \quad \nu A + B + \lambda C, \quad \mu A + \lambda B + C \ldots (71).$$

Suppose the circle given by an equation between sine coordinates, or in the form

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = 0 \ldots\ldots\ldots\ldots (72).$$

The cosine system of coordinates for the pole is given by

$$\epsilon = \mathrm{A},\ \mathrm{B},\ \mathrm{C} \ldots\ldots\ldots\ldots (73),$$

and the sine system by

$$e = \mathfrak{a}\mathrm{A} + \mathfrak{h}\mathrm{B} + \mathfrak{g}\mathrm{C}, \quad \mathfrak{h}\mathrm{A} + \mathfrak{b}\mathrm{B} + \mathfrak{f}\mathrm{C}, \quad \mathfrak{g}\mathrm{A} + \mathfrak{f}\mathrm{B} + \mathfrak{c}\mathrm{C} \ldots (74).$$

It is hardly necessary to observe, that if

$$A\xi + B\eta + C\zeta = 0 \quad . \quad . \quad . \quad . \quad (75),$$

$$Ax + By + Cz = 0 \quad . \quad . \quad . \quad . \quad (76),$$

represent the same great circle,

$$\mathrm{A} : \mathrm{B} : \mathrm{C} = A + \nu B + \mu C : \nu A + B + \lambda C : \mu A + \lambda B + C \ldots (77),$$

$$A : B : C = \mathfrak{a}\mathrm{A} + \mathfrak{h}\mathrm{B} + \mathfrak{g}\mathrm{C} : \mathfrak{h}\mathrm{A} + \mathfrak{b}\mathrm{B} + \mathfrak{f}\mathrm{C} : \mathfrak{g}\mathrm{A} + \mathfrak{f}\mathrm{B} + \mathfrak{c}\mathrm{C} \ldots (78).$$

Inclination of two great Circles.

Let the equations of these be

$$\begin{cases} A\xi + B\eta + C\zeta = 0 \quad . \quad . \quad . \quad (79), \\ \text{or } \mathrm{A}x + \mathrm{B}y + \mathrm{C}z = 0 \quad . \quad . \quad . \quad (80), \end{cases}$$

$$\begin{cases} A'\xi + B'\eta + C'\zeta = 0 \quad . \quad . \quad . \quad (81), \\ \text{or } \mathrm{A}'x + \mathrm{B}'y + \mathrm{C}'z = 0 \quad . \quad . \quad . \quad (82), \end{cases}$$

and let e, ϵ, have the same values as above, and e', ϵ', corresponding ones. To obtain the inclination of the two circles, we have only, in the formulæ given above for the distance of two points, to change p, p', ω, ω', into $e, e', \epsilon, \epsilon'$.

The distance of a point from a given circle may be found with equal facility; for this is evidently the compliment of the distance of the point from the pole of the circle. In like manner we may find the condition that two great circles intersect at right angles, &c.

There are evidently a whole class of formulæ, not by any means peculiar to the present system of coordinates, such as

$$Ax + By + Cz - s\,.\,(A'x + B'y + C'z) \quad . \quad . \quad (83),$$

for the equation of a great circle subjected to pass through the points of intersection of

$$Ax + By + Cz = 0, \qquad A'x + B'y + C'z = 0.$$

Again,

$$\begin{vmatrix} x, & y, & z \\ a, & b, & c \\ a', & b', & c' \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad (84),$$

for the equation of the great circle which passes through the points given by the sine systems $a : b : c$ and $a' : b' : c'$, &c., and which are obtained so easily that it is not worth while writing down any more of them.

Transformation of Coordinates.

Let X_1, Y_1, Z_1, be the new points of reference, and suppose X_1 is given by the conjoint systems $e = a, b, c$, $\epsilon = \alpha, \beta, \gamma$; and similarly Y_1, Z_1, by the analogous systems $e', \epsilon' - e'', \epsilon''$.

Suppose, as before, P is given by one of the systems ω, p; and let ω_1, p_1 be the new systems which determine the position of P with reference to X_1, Y_1, Z_1.

In the first place, λ_1, μ_1, ν_1, are given by the formulæ

$$\lambda_1 = \frac{W(e', e'', \mathrm{q})}{\sqrt{\{W(e', e', \mathrm{q})\, W(e'', e'', \mathrm{q})\}}} = \frac{W(\epsilon', \epsilon'', \mathfrak{q})}{\sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\, W(\epsilon'', \epsilon'', \mathfrak{q})\}}} \quad \ldots . (85),$$

$$\mu_1 = \frac{W(e'', e, \mathrm{q})}{\sqrt{\{W(e'', e'', \mathrm{q})\, W(e, e, \mathrm{q})\}}} = \frac{W(\epsilon'', \epsilon, \mathfrak{q})}{\sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\, W(\epsilon, \epsilon, \mathfrak{q})\}}},$$

$$\nu_1 = \frac{W(e, e', \mathrm{q})}{\sqrt{\{W(e, e, \mathrm{q})\, W(e', e', \mathrm{q})\}}} = \frac{W(\epsilon, \epsilon', \mathfrak{q})}{\sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\, W(\epsilon', \epsilon', \mathfrak{q})\}}}.$$

The system ω_1 is evidently given immediately by

$$\xi_1 : \eta_1 : \zeta_1 = \frac{W(e, p, \mathrm{q})}{\sqrt{\{W(e, e, \mathrm{q})\}}} : \frac{W(e', p, \mathrm{q})}{\sqrt{\{W(e', e', \mathrm{q})\}}} : \frac{W(e'', p, \mathrm{q})}{\sqrt{\{W(e'', e'', \mathrm{q})\}}} \ldots (86)$$

$$= \frac{W(\epsilon, \omega, \mathfrak{q})}{\sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\}}} : \frac{W(\epsilon', \omega, \mathfrak{q})}{\sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\}}} : \frac{W(\epsilon'', \omega, \mathfrak{q})}{\sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\}}} \ldots (87).$$

And from these we may obtain the system p_1, by means of the formulæ

$$x_1 : y_1 : z_1 = \mathfrak{a}_1\xi_1 + \mathfrak{h}_1\eta_1 + \mathfrak{g}_1\zeta_1 : \mathfrak{h}_1\xi_1 + \mathfrak{b}_1\eta_1 + \mathfrak{f}_1\zeta_1 : \mathfrak{g}_1\xi_1 + \mathfrak{f}_1\eta_1 + \mathfrak{c}_1\zeta_1 \ldots (88).$$

This requires some further development however. We must in the first place form the system $\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{c}_1, \mathfrak{f}_1, \mathfrak{g}_1, \mathfrak{h}_1$: this is done immediately from the formulæ of Sect. 2, and we have

$$\mathfrak{a}_1 = \frac{W(\overline{e'e''}, \overline{e'e''}, \mathfrak{q})}{W(e', e', \mathrm{q})\, W(e'', e'', \mathrm{q})} = \frac{k\, W(\overline{\epsilon'\epsilon''}, \overline{\epsilon'\epsilon''}, \mathrm{q})}{W(\epsilon', \epsilon', \mathfrak{q})\, W(\epsilon'', \epsilon'', \mathfrak{q})} \ldots (89),$$

$$\mathfrak{b}_1 = \frac{W(\overline{e''e}, \overline{e''e}, \mathfrak{q})}{W(e'', e'', \mathrm{q})\, W(e'', e, \mathrm{q})} = \frac{k\, W(\overline{\epsilon''\epsilon}, \overline{\epsilon''\epsilon}, \mathrm{q})}{W(\epsilon'', \epsilon'', \mathfrak{q})\, W(\epsilon, \epsilon, \mathfrak{q})},$$

$$\mathfrak{e}_1 = \frac{W(\overline{ee'}, \overline{ee'}, \mathfrak{q})}{W(e, e, q)\, W(e', e', q)} = \frac{k W(\overline{\epsilon\epsilon'}, \overline{\epsilon\epsilon'}, q)}{W(\epsilon, \epsilon, \mathfrak{q})\, W(\epsilon', \epsilon', \mathfrak{q})},$$

$$\mathfrak{f}_1 = \frac{W(\overline{e''e}, \overline{ee'}, \mathfrak{q})}{W(e, e, q)\, \sqrt{\{W(e', e', q)\, W(e'', e'', q)\}}}$$

$$= \frac{k W(\overline{\epsilon''\epsilon}, \overline{\epsilon\epsilon'}, q)}{W(\epsilon, \epsilon, \mathfrak{q}) \sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\, W(\epsilon'', \epsilon'', \mathfrak{q})\}}},$$

$$\mathfrak{g}_1 = \frac{W(\overline{ee'}, \overline{e'e''}, \mathfrak{q})}{W(e', e', q)\, \sqrt{\{W(e'', e'', q)\, W(e, e, q)\}}}$$

$$= \frac{k W(\overline{\epsilon\epsilon'}, \overline{\epsilon'\epsilon''}, q)}{W(\epsilon', \epsilon', \mathfrak{q})\, \sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\, W(\epsilon, \epsilon, \mathfrak{q})\}}},$$

$$\mathfrak{h}_1 = \frac{W(\overline{e'e''}, \overline{e''e}, \mathfrak{q})}{W(e'', e'', q)\, \sqrt{\{W(e, e, q)\, W(e', e', q)\}}}$$

$$= \frac{k W(\overline{\epsilon'\epsilon''}, \overline{\epsilon''\epsilon}, q)}{W(\epsilon'', \epsilon'', \mathfrak{q})\, \sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\, W(\epsilon', \epsilon', \mathfrak{q})\}}}.$$

$$x_1 : y_1 : z_1 = \sqrt{\{W(e, e, q)\}}\, \{W(e, p, q)\, W(\overline{e'e''}, \overline{e'e''}, \mathfrak{q})$$
$$+ W(e', p, q)\, W(\overline{e'e''}, \overline{e''e}, \mathfrak{q}) + W(e'', p, \mathfrak{q})\, W(\overline{e'e''}, \overline{ee'}, \mathfrak{q})\} \ldots (90),$$
$$: \sqrt{\{W(e', e', q)\}}\, \{W(e, p, q)\, W(\overline{e''e}, \overline{e'e''}, \mathfrak{q})$$
$$+ W(e', p, q)\, W(\overline{e''e}, \overline{e''e}, \mathfrak{q}) + W(e'', p, q)\, W(\overline{e''e}, \overline{ee'}, q)\}$$
$$: \sqrt{\{W(e'', e'', q)\}}\, \{W(e, p, q)\, W(\overline{ee'}, \overline{e'e''}, \mathfrak{q})$$
$$+ W(e', p\ q)\, W(\overline{ee'}, \overline{e''e}, \mathfrak{q}) + W(e'', p, q)\, W(\overline{ee'}, \overline{ee'}, \mathfrak{q})\};$$

which may be reduced to the very simple form

$$x_1 : y_1 : z_1 = \sqrt{\{W(e, e, q)\}}\, W(\overline{e'e''}, \omega, \mathfrak{q}) \ldots . (91),$$
$$: \sqrt{\{W(e', e', q)\}}\, W(\overline{e''e}, \omega, \mathfrak{q}),$$
$$: \sqrt{\{W(e'', e'', q)\}}\, W(\overline{ee'}, \omega, \mathfrak{q}).$$

And in like manner we obtain

$$x_1 : y_1 : z_1 = \sqrt{\{W(\epsilon, \epsilon, \mathfrak{q})\}}\, W(\overline{\epsilon'\epsilon''}, p, q) \ldots . (92),$$
$$: \sqrt{\{W(\epsilon', \epsilon', \mathfrak{q})\}}\, W(\overline{\epsilon''\epsilon}, p, q),$$
$$: \sqrt{\{W(\epsilon'', \epsilon'', \mathfrak{q})\}}\, W(\overline{\epsilon\epsilon'}, p, q).$$

It will be as well to indicate the steps of this reduction. Consider the quantity in { } in the first line of the equation which gives the ratios $x_1 : y_1 : z_1$; and suppose for a moment $\overline{e'e''} = l, m, n$, &c., selecting the portion of the expression which is multiplied by $\mathfrak{a}$, this is

$$\mathfrak{a}l\,.\,\{l(a\xi + b\eta + c\zeta) + l'(a'\xi + b'\eta + c'\zeta) + l''(a''\xi + b''\eta + c''\zeta)\}.$$

Or, since

$$la + l'a' + l''a'' = \underline{\overline{ee'e''}},\quad lb + l'b' + l''b'' = 0,\quad lc + l'c' + l''c'' = 0,$$

this reduces itself to $\underline{\overline{ee'e''}}$, $\mathfrak{a}l\xi$, which is a term of

$$\underline{\overline{ee'e''}}\, W(\underline{\overline{e'e''}}, \omega, \mathfrak{q});$$

and by comparing the remaining terms in the same manner, it would be seen that the whole reduces itself to

$$\underline{\overline{ee'e''}}\, W(\underline{\overline{e'e''}}, \omega, \mathfrak{q});$$

whence the formulæ in question.

The formulæ (86), (87), (91), (92), completely resolve the problem of the transformation of coordinates; they determine respectively p_1 from p or ω, ω_1 from p or ω.

To complete the present part of the subject we may add the following formulæ.

Suppose

$$\begin{aligned} x_1 : y_1 : z_1 &= \mathrm{a}x_1 + \mathrm{a}'y_1 + \mathrm{a}''z_1 \;\ldots\ldots\ldots\ldots \text{(93)},\\ &: \mathrm{b}x_1 + \mathrm{b}'y_1 + \mathrm{b}''y_1,\\ &: \mathrm{c}x_1 + \mathrm{c}'y_1 + \mathrm{c}''z_1,\end{aligned}$$

which we see from the preceding formulæ is the form of the relation between the systems p_1 and p. And suppose, as before, λ_1, μ_1, ν_1 are the cosines of the distances of the new points of reference X_1, Y_1, Z_1.

We can immediately determine the relations that must exist between these coefficients, in order that they may actually denote such a transformation. For this purpose write

$$\begin{aligned} \mathrm{a}, \mathrm{b}, \mathrm{c} &= j \;\ldots\ldots\ldots\ldots\ldots \text{(94)},\\ \mathrm{a}', \mathrm{b}', \mathrm{c}' &= j',\\ \mathrm{a}'', \mathrm{b}'', \mathrm{c}'' &= j''.\end{aligned}$$

Then the distance between the point P and any other point P' is given by the formula

$$\begin{aligned}\cos\delta &= \frac{W(p, p', \mathfrak{q})}{\sqrt{\{W(p, p, \mathfrak{q})\,W(p', p', \mathfrak{q})\}}}\\ &= \frac{W(p_1, p_1', \Theta)}{\sqrt{\{W(p_1, p_1, \Theta)\,W(p_1, p_1', \Theta)\}}} \;\ldots\ldots \text{(95)},\end{aligned}$$

where $\Theta = W(j, j, \mathrm{q}),\ W(j', j', \mathrm{q}),\ W(j'', j'', \mathrm{q}),$

$$W(j', j'', \mathrm{q}),\ W(j'', j, \mathrm{q}),\ W(j, j', \mathrm{q}) \ldots\ldots (96).$$

But we must evidently have

$$\cos\delta = \frac{W(p_1, p_1', \mathrm{q}_1)}{\sqrt{\{W(p_1, p_1, \mathrm{q}_1)\, W(p_1', p_1', \mathrm{q}_1)\}}} \ldots\ldots (97),$$

or the quantities Θ must be proportional to the quantities q, i.e.

$$W(j, j, \mathrm{q}) : W(j', j', \mathrm{q}) : W(j'', j'', \mathrm{q}) : W(j', j'', \mathrm{q})$$
$$: W(j'', j, \mathrm{q}) : W(j'', j, \mathrm{q}) \ldots\ldots (98),$$
$$= 1 : 1 : 1 : \lambda_1 : \mu_1 : \nu_1.$$

And in precisely the same manner, if instead of x, y, z, x_1, y_1, z_1, in the above formulæ, we had had $\xi, \eta, \zeta : \xi_1, \eta_1, \zeta_1$, the result would have been

$$W(j, j, \mathfrak{q}) : W(j', j', \mathfrak{q}) : W(j'', j'', \mathfrak{q}) : W(j', j'', \mathfrak{q})$$
$$: W(j'', j, \mathfrak{q}) : W(j'', j, \mathfrak{q})$$
$$= \mathfrak{a} : \mathfrak{b} : \mathfrak{c} : \mathfrak{f} : \mathfrak{g} : \mathfrak{h} \ldots\ldots\ldots\ldots (99).$$

It is hardly necessary to remark, that throughout the preceding formulæ an expression, such as $W(p, p', \mathrm{q})$, is proportional to either of the quantities

$$x\xi' + y\eta' + z\zeta' \text{ or } x'\xi + y'\eta + z'\zeta,$$

and may be changed into one of these multiplied by an arbitrary constant; which may be always made to disappear by a corresponding change in another quantity of the same form: thus, for instance,

$$\frac{W(p, p', \mathrm{q})}{W(p, p, \mathrm{q})} = \frac{x'\xi + y'\eta + z'\zeta}{x\xi + y\eta + z\zeta} \ldots\ldots (100);$$

but these forms being unsymmetrical, it is better in general not to introduce them.

All the preceding expressions simplify exceedingly, reducing themselves in fact to the ordinary formulæ for the transformation of rectangular coordinates in Geometry of three dimensions, for the case where the triangle XYZ has its sides and angles right angles. As this presents no difficulty, I shall not enter upon it at present.

NOTE ON INDUCED MAGNETISM IN A PLATE.

By William Thomson, B.A., Fellow of St Peter's College.

If a plate of soft iron be submitted to the action of a magnet of any kind, it immediately becomes magnetized "by induction;" and the effects of this are exhibited in the attraction or repulsion it exercises upon small magnetic bodies in its neighbourhood. The determination of these effects, from the elementary laws of magnetic induction, is a problem of considerable practical interest. In the case of a plate bounded by infinite parallel planes, I have succeeded in obtaining a complete solution of a very simple nature, by means of a principle which will be developed in a future paper. The object of the present note is to compare this solution with a formula given by Green in his *Essay on Electricity and Magnetism*, as an approximate result, but which appears to be inadmissible.

Let the influencing magnet, which may be of any form and size, and magnetized in any manner, be denoted by Q; and let us suppose it to be held *behind* the plate of soft iron. The solution which I have obtained enables us to find the total magnetic action on a point, P, situated in any position, either within or without the plate; but at present I shall only state the result when P is *before* the plate. In this case the actual magnetic effect on P may be produced by supposing Q and the plate to be removed, and a certain imaginary series of magnets Q', Q_1, Q_2, &c. to be substituted, the system being constructed thus. Each of the imaginary magnets is equal and similar to Q, and similarly magnetized; Q' occupies the place of Q, and the others are similarly placed behind it, along a line perpendicular to the plate, the distance between corresponding points of each consecutive pair being equal to twice the thickness of the plate. The intensities of the successive magnets decrease in a geometrical progression, of which the common ratio is m^2, (a quantity measuring the inductive capacity for magnetism of the plate,) commencing with that of Q', which is equal to $1 - m^2$, if the intensity of Q be unity. It is hardly necessary to point out the analogy between this and the corresponding result in optics, in which the illumination produced through a plate of glass, by a candle, is found to be due to the candle itself, with diminished brightness, and to a row of images behind it, with intensities decreasing in a geometrical progression, which arise from successive internal reflections.

If the iron plate be infinitely thin, all the *images*, Q_1, Q_2, &c. will coincide with Q'; and, since the sum of their intensities is unity, the total effect will be the same as that of Q, which will therefore be unaffected by the interposition of the screen. The same will be the case if the distance of Q be infinitely great, and the thickness of the screen finite; but in this case, at least as far as the present result can shew us, the dimensions of the planes which bound the plate must be infinitely great compared with the distance of Q.

The result which I have stated is applicable also to the imaginary case in which, instead of being a magnet, Q is a mass of positive or negative magnetism.* Thus, let Q be a unit of positive magnetism collected in a point, which case is investigated by Green. To express the action analytically, let Q be taken as origin of coordinates, a line perpendicular to the plate as axis of x, and the plane through this line, and P, as plane of (x, y). Then denoting by a the thickness of the plate, and considering Q as a positive unit of matter, we shall have, for the total potential at P, due to Q and the plate,

$$F = (1 - m^2)\left\{\frac{1}{(x^2 + y^2)^{\frac{1}{2}}} + \frac{m^2}{\{(x + 2a)^2 + y^2\}^{\frac{1}{2}}} + \frac{m^4}{\{(x + 4a)^2 + y^2\}^{\frac{1}{2}}} + \&c.\right\} \quad \ldots\ldots (1).$$

For all magnetic bodies m is between 0 and 1, the former limit being its value when the inductive capacity for magnetism is nothing, and the latter being never attained, though it is approached in such bodies as iron, of which the inductive capacity is great. In the extreme case of $m = 1$, the laws of induction in a magnetic body degenerate into those of electrical equilibrium on the surface of a conductor of electricity. If in the expression for F we put $m = 1$, one of the factors vanishes and the other becomes infinite, but the ultimate value of the product is nothing, which shews that the effect of the plate is to destroy all action behind it. This we know to be the case when an infinite conducting screen of any form is placed before an electrified body.

* This expression does not imply any hypothesis of a magnetic matter or of a fluid or fluids, but it is merely used for brevity in consequence of the principle established by Coulomb, Poisson, and Ampère, that the action of a magnetized body of any kind, or of a collection of electric "closed currents," may always be represented by an imaginary positive and negative distribution of matter, of which the whole mass is algebraically nothing. By an element of positive or negative magnitude, we merely mean a portion of this imagined matter.

In the case when the plate is of iron, the value of m is nearly unity. Hence, as the series is multiplied by $1 - m^2$, it might be imagined that, if we "neglect small quantities of the order $(1 - g)$ compared with those which are retained," ($1 - g$ being, in Green's notation, a quantity of the same order as $1 - m$), an approximate result would be obtained by putting $m = 1$ in the successive terms of the series within the vinculum. And it is thus that Green, having, in the investigation, neglected quantities multiplied by $(1 - g)^2$, arrives at the result,

$$F = \frac{4(1-g)}{3}\left\{\frac{1}{(x^2+y)^{\frac{1}{2}}} + \frac{1}{\{(x+2a)^2+y^2\}^{\frac{1}{2}}} + \frac{1}{\{(x+4a)^2+y^2\}^{\frac{1}{2}}} + \&c.\right\}$$

As, however, this series has an infinite sum, it is clear that no value of m can be sufficiently near to unity to render the approximation admissible. If instead of Q we were to substitute a magnet, or any collection of positive and negative particles, such that the sum of the masses is zero, the series for the potential, deduced from Green's expression, would converge: and the same remark is applicable to the series which would be found for the *attraction* of the system on a point beyond the screen, even when Q is a positive point, by differentiating the expression for F. Notwithstanding this, the approximation is still inadmissible; since, if we expand the rigorous expression in either case in ascending powers $(1 - m)$, we find that, though the first term is finite, the coefficients of all the terms which follow it are infinite.

Although the method by which I obtained the rigorous solution is quite distinct from that followed by Green, being independent of any mathematical process, it may be satisfactory to shew that the result can be deduced from his own analysis, and even with greater ease than his solution is obtained, after making unnecessary approximation.

By a very remarkable investigation, in which he extends Laplace's well-known analysis for spherical coordinates to the case when the radius of the sphere becomes infinite, Green arrives (*Essay on Electricity*, p. 64) at the following expression for the total potential at P, due to the positive unit of matter Q, and to the interposed plate, before making any approximation:

$$F = \frac{8}{\pi}(1-g)(1+2g)\int_0^\infty \frac{d\gamma\, \epsilon^{-\gamma x}}{(2+g)^2 - 9g^2\epsilon^{-2\gamma a}}\int_0^1 \frac{d\beta}{(1-\beta^2)^{\frac{1}{2}}}\cos(\beta\gamma y).$$

Let $m = \frac{3g}{2+g}$. Then we have, by expansion, and by changing the order of the integration,

$$F = \frac{2}{\pi}(1-m^2)\int_0^1 \frac{d\beta}{(1-\beta^2)^{\frac{1}{2}}}$$

$$\int_0^\infty d\gamma \,.\, \epsilon^{-\gamma x}(1 + m^2\epsilon^{-2\gamma a} + m^4\epsilon^{-4\gamma a} + \&c.)\cos(\beta\gamma y)$$

$$= \frac{2}{\pi}(1-m^2)\int_0^1 \frac{d\beta}{(1-\beta^2)^{\frac{1}{2}}}$$

$$\left\{\frac{x}{x^2+\beta^2y^2} + \frac{m^2(x+2a)}{(x+2a)^2+\beta^2y^2} + \frac{m^4(x+4a)}{(x+4a)^2+\beta^2y^2} + \&c.\right\}$$

$$= \frac{2}{\pi}(1-m^2)\,\Sigma\int_0^{\frac{1}{2}\pi}\frac{m^{2i}x_i\,d\theta}{x_i^2+y^2\sin^2\theta}, \quad \text{where } x_i = x + 2ia,$$

$$= (1-m^2)\,\Sigma\,\frac{m^{2i}}{(x_i^2+y^2)^{\frac{1}{2}}},$$

which agrees with the expression given above.

St. Peter's College, Oct. 14th, 1845.

ON THE VARIATION OF ELEMENTS IN THE PLANETARY THEORY.

By Hugh Blackburn, B.A. Trinity College.

The six quantities which determine the form and position of a planet's elliptic orbit and the planet's place at the period from which time is measured, are called the elements of its orbit. When they are known, Kepler's second and third laws are sufficient to determine the planet's place at any subsequent period. In nature, owing to the action of the planets on one another, their orbits are not accurately fixed ellipses; but the disturbing force being small, they may conveniently be represented as moving in variable ellipses, whose elements at any instant are determined by the condition, that the actual motion shall be the same as if they moved undisturbed in those ellipses. This method is naturally suggested by observation, but was first treated analytically by Lagrange.

Let xyz, $x'y'z'$, be the coordinates, with reference to three rectangular axes fixed in space, of the disturbed and disturbing planets respectively: and let

$$R = \frac{m'}{\{(x-x')^2+(y-y')^2+(z-z')^2\}^{\frac{1}{2}}} - \frac{m'(xx'+yy'+zz')}{(x'^2+y'^2+z'^2)^{\frac{3}{2}}}.$$

Then the equations of the disturbed planet's motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{\mu x}{r^3} &= \frac{dR}{dx} \\ \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} &= \frac{dR}{dy} \\ \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} &= \frac{dR}{dz} \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{(I)},$$

which coincide with the equations of undisturbed motion when we put $R = 0$. The six arbitrary constants introduced by the integration of these equations, when $R = 0$, are the elements or functions of them, and the variable elements in the case of disturbed motion are to be found from the six relations given by making the expressions for $x, y, z, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$, the same as if the disturbing force were to cease. If the values of x, y, z, x', y', z', in terms of t and the elements, be substituted in R, we have the following remarkable theorem. The variations (or differential coefficients with respect to t) of all the elements may be rigorously expressed as linear functions of the partial differential coefficients of R with respect to the elements, multiplied by functions of the elements themselves not involving t explicitly. This was proved from general considerations by Lagrange, and afterwards extended to the case of any mechanical problem, nearly at the same time that Laplace independently deduced similar expressions for the variations from the consideration of elliptic motion.*

It is well, for the sake of clearness, to distinguish three kinds of change which the orbit of the planet may undergo: the form and size of the ellipse may be altered; the orbit (or rather the lines of perihelion and epoch) may revolve in its plane; and the plane of the orbit may revolve about the radius vector of the planet as instantaneous axis. The two former are due to the resolved part of the disturbing force in the plane of the orbit, and the values of the variations of the elements corresponding will be the same, whether the force perpendicular to the plane of the orbit be taken account of or not. This will apply to the axis major and eccentricity, and also to the longitude of perihelion and epoch, if measured from a line which remains fixed relatively to the plane

* See Lagrange in the *Memoirs of the Institute* for 1808, 1809, and *Mech. Analyt.* last edit. sect. V. and VII. Part. II.; also Poisson in the *Journal de l'Ec. Polytech.* Cahier XVI.

of the orbit. In this case the expressions deduced in Airy's Tracts* will still be rigorously true, when the motion of the plane of the orbit is considered. It is more convenient, however, to measure these angles from the line of nodes, when they will be affected by the motion of the plane, and consequently the same expressions will not apply. The necessary modifications for this way of measuring the angles will be given hereafter. The motion of the plane of the orbit, and the corresponding variations of node and inclination, which are due to the disturbing force perpendicular to the plane of the orbit, are not considered in Airy's Tract. In Pratt's *Mechanics* the rigorous expressions are given, but the investigation of them is only approximate, and seems no simpler than a rigorous method.

These few pages will, I hope, be found useful as a supplement to Airy's Tract, and a substitute for the investigation in Pratt.

For distinctness, the plane of the disturbed planet's orbit is supposed to intersect the planes of xy and yz between the positive axes, and the positive direction of measuring angles in the plane of the orbit is from the intersection with xy to that with yz.

Let l, m, n, be the direction-cosines of the plane of the orbit, and Ω the longitude of node measured from the axis of x in the plane of xy, i the inclination of the plane of the orbit to that of xy, so that we have

$$l = \sin i \sin \Omega, \quad m = -\sin i \cos \Omega, \quad n = \cos i.$$

Let H be twice the area described on the plane of the orbit in a unit of time, and therefore lH, mH, nH, the corresponding areas on the coordinate planes. Let also ξ, η, ζ, be the coordinates of the planet referred to three rectangular axes, so chosen, that the plane of ξ, η, coincides with the plane of the orbit at the time t, so that $\zeta = 0$ at that instant. Then the equation to the plane of the orbit is

$$lx + my + nz = 0,$$

where $$l^2 + m^2 + n^2 = 1.$$

Differentiating these two equations, and remembering that $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, are to be expressed in the same way as if the motion were not disturbed, we have

* The investigation of the variation of perihelion given by Mr. O'Brien, vol. I., p. 165, of this Journal, is preferable to that in Airy's *Tracts*.

$$l\frac{dx}{dt}+m\frac{dy}{dt}+n\frac{dz}{dt}=0\ \ldots\ldots\ldots\ldots(1),$$

$$x\frac{dl}{dt}+y\frac{dm}{dt}+z\frac{dn}{dt}=0\ \ldots\ldots\ldots\ldots(2),$$

$$l\frac{dl}{dt}+m\frac{dm}{dt}+n\frac{dn}{dt}=0\ \ldots\ldots\ldots\ldots(3).$$

And differentiating (1) again, we have by equations (I)

$$\frac{dl}{dt}\,\frac{dx}{dt}+\frac{dm}{dt}\,\frac{dy}{dt}+\frac{dx}{dt}\,\frac{dz}{dt}=-\left(l\frac{d^2x}{dt^2}+m\frac{d^2y}{dt^2}+n\frac{d^2z}{dt^2}\right)$$

$$=-\left(l\frac{dR}{dx}+m\frac{dR}{dy}+n\frac{dR}{dz}\right)=-\frac{dR}{d\zeta}=-S\ \ldots\ (4),$$

where S is the force perpendicular to the plane of the orbit.

Combining (2) and (3) with this last, we get

$$\frac{\frac{dl}{dt}}{mz-ny}=\frac{\frac{dm}{dt}}{nx-lz}=\frac{\frac{dn}{dt}}{ly-mx}=\frac{-\frac{dR}{d\zeta}}{H}=\frac{-S}{H}\ \ldots\ (5),$$

since $y\frac{dz}{dt}-z\frac{dy}{dt}=lH,\ z\frac{dx}{dt}-x\frac{dz}{dt}=mH,\ x\frac{dy}{dt}-y\frac{dx}{dt}=nH.$

Also, since the velocity perpendicular to the plane of the instantaneous orbit is always zero,

$$\frac{d\zeta}{dt}=\frac{d\zeta}{di}\,\frac{di}{dt}+\frac{d\zeta}{d\Omega}\,\frac{d\Omega}{dt}=0\ \ldots\ldots\ldots\ (6);$$

and $$\frac{d\zeta}{d\Omega}=l\frac{dx}{d\Omega}+m\frac{dy}{d\Omega}+n\frac{dz}{d\Omega}=-\left(x\frac{dl}{d\Omega}+y\frac{dm}{d\Omega}+z\frac{dn}{d\Omega}\right)$$

$$=-(x\cos\Omega+y\sin\Omega)\sin i\ \ldots\ (7),$$

$$\frac{d\zeta}{di}=-\left(x\frac{dl}{di}+y\frac{dm}{di}+z\frac{dn}{di}\right)=-\frac{x}{n}\left(n\frac{dl}{di}-l\frac{dn}{di}\right)-\frac{y}{n}\left(n\frac{dm}{di}-m\frac{dn}{di}\right)$$

$$=-(x\sin\Omega-y\cos\Omega)\sec i\ \ldots\ (8).$$

Now, from (5) and (7),

$$\frac{dn}{dt}=-\sin i\,\frac{di}{dt}=-\frac{S}{H}(x\cos\Omega+y\sin\Omega)\sin i,$$

$$\therefore\ \frac{di}{dt}=\frac{S}{H}(x\cos\Omega+y\sin\Omega)=-\frac{1}{H\sin i}\,\frac{dR}{d\zeta}\cdot\frac{d\zeta}{d\Omega}.$$

And, from (6) and (8),

$$\frac{d\Omega}{dt}=\frac{1}{H\sin i}\,\frac{dR}{d\zeta}\,\frac{d\zeta}{di}=-\frac{S}{H\sin i\cos i}\,(x\sin\Omega-y\cos\Omega).$$

Hence, if the axis of ξ be made to coincide with the line of nodes at time t, and if r and θ be the polar co-ordinates in the plane of the orbit, we shall have the various expressions,

$$\left.\begin{aligned}\frac{di}{dt}&=\frac{S.\xi}{H}=\frac{S.r\cos\theta}{H}=-\frac{1}{H\sin i}\,\frac{dR}{d\zeta}\,\frac{d\zeta}{d\Omega}\\ \frac{d\Omega}{dt}&=\frac{S.\eta}{H\sin i}=\frac{S.r\sin\theta}{H\sin i}=\frac{1}{H\sin i}\,\frac{dR}{d\zeta}\,\frac{d\zeta}{di}\end{aligned}\right\}\ldots\text{(A)}.$$

As R is to be expressed in terms of t and the elements, the latter part of these expressions must be transformed before they can be of much use, but the method of measuring the angles in the plane of the orbit (the longitude of perihelion and epoch) must first be fixed. One way is to suppose the angles measured from a *fixed line* in the plane of the orbit, which implies that the angles in that plane are not affected by its rotation about any line in itself, but only by rotation of the orbit about the normal to its plane. On this supposition, if the longitude of node is to be varied alone, the plane of the orbit must revolve about a line in itself at right angles to the line of nodes, and, if the inclination alone, about the line of nodes. The variation therefore of R by a variation of Ω or i alone is produced by a motion of the planet perpendicular to the plane of its orbit, and we shall therefore have

$$\frac{dR}{di}=\frac{dR}{d\zeta}\,\frac{d\zeta}{di},\quad \frac{dR}{d\Omega}=\frac{dR}{d\zeta}\,\frac{d\zeta}{d\Omega},$$

and the expressions (A) become

$$\frac{di}{dt}=-\frac{1}{H\sin i}\,\frac{dR}{d\Omega},\quad \frac{d\Omega}{dt}=\frac{1}{H\sin i}\,\frac{dR}{di}\;\ldots\ldots(a).$$

These are the simplest expressions for the variations of Ω and i; but for the purpose of determining the planet's place the mode of measuring the angles implied is not the best, for, when the plane of the orbit changes, it becomes necessary to have some means of finding the position of the fixed line, before the angles can be measured from it. For this purpose let ϑ be the angle between the fixed line and the line of nodes. Then to find ϑ we have the relation

$$d\vartheta=-\cos i\,d\Omega,$$

or

$$\vartheta=C-\int\cos i\,d\Omega.$$

The angle ϑ is sometimes called the longitude of the node in the orbit. When it is determined, the position of the fixed line is known, and then the planet's place can be found by measuring the angles from it.

The better method is at once to measure the angles from the line of nodes, or, which gives a simpler result, to measure the longitude of perihelion from the line of nodes, and the epoch, which will in this case be the value of the mean anomaly (instead of the mean longitude) when $t = 0$, from perihelion. Let ω be the longitude of perihelion, and c the epoch thus measured.

If now Ω is to be made to vary alone, the orbit must turn about an axis perpendicular to the plane xy, so as to preserve i, ω, and c unchanged. This motion is the same as rotation about an axis in its plane perpendicular to the line of nodes through an angle $d\Omega \sin i$, combined with rotation about the normal to its plane through an angle $d\Omega \cos i$.

Now $$\frac{dR}{d\Omega} = \frac{dR}{d\xi}\frac{d\xi}{d\Omega} + \frac{dR}{d\eta}\frac{d\eta}{d\Omega} + \frac{dR}{d\zeta}\frac{d\zeta}{d\Omega},$$

and $\left(\frac{dR}{d\xi}\frac{d\xi}{d\Omega} + \frac{dR}{d\eta}\frac{d\eta}{d\Omega}\right) d\Omega$ is the variation of R in consequence of the rotation of the orbit through the angle $d\Omega \cos i$ about the normal to its plane. But the only element affected by rotation about the normal is ω; therefore the change of R by rotation through the angle $d\Omega \cos i$ is $= \frac{dR}{d\omega} d\Omega \cos i$;

therefore $$\frac{dR}{d\xi}\frac{d\xi}{d\Omega} + \frac{dR}{d\eta}\frac{d\eta}{d\Omega} = \frac{dR}{d\omega}\cos i;$$

and therefore $$\frac{dR}{d\zeta}\frac{d\zeta}{d\Omega} = \frac{dR}{d\Omega} - \frac{dR}{d\omega}\cos i.$$

The value of $\frac{dR}{d\zeta}\frac{d\zeta}{di}$ is the same as before, since the motion of the plane of the orbit, when i alone varies, is the same. The expressions (A) in this case will therefore become

$$\left.\begin{aligned} \frac{di}{dt} &= -\frac{1}{H\sin i}\frac{dR}{d\Omega} + \frac{\cos i}{H\sin i}\frac{dR}{d\omega} \\ \frac{d\Omega}{dt} &= \frac{1}{H\sin i}\frac{dR}{di} \end{aligned}\right\} \quad \ldots\ldots (b).$$

If the epoch be measured from the line of nodes, it is at once

evident that (calling this epoch ε) $\frac{dR}{d\omega}$ must be replaced by $\frac{dR}{d\varepsilon} + \frac{dR}{d\omega}$, both ε and ω being changed by rotation about the normal to the plane of the orbit. We have also the relation $\varepsilon = c + \omega$, which would give the same result by transformation. The expressions (b) will thus become

$$\left.\begin{aligned} \frac{di}{dt} &= -\frac{1}{H \sin i}\,\frac{dR}{d\Omega} + \frac{\cos i}{H \sin i}\left(\frac{dR}{d\varepsilon} + \frac{dR}{d\omega}\right) \\ \frac{d\Omega}{dt} &= \frac{1}{H \sin i}\,\frac{dR}{di} \end{aligned}\right\} \ldots .\ (b').$$

It should be observed, that the variations of longitude of perihelion and epoch will also be differently expressed in the two cases above given. If ϖ and ϵ be the longitude of perihelion and epoch in the first case, the values of $\frac{d\varpi}{dt}$ and $\frac{d\epsilon}{dt}$ will be those given in Airy and Pratt,* since the angles are altered only by rotation of the orbit in its plane.

If we take the epoch c, measured from perihelion, the value of $\frac{dc}{dt}$, which $= \frac{d\epsilon}{dt} - \frac{d\varpi}{dt}$, will be the same, whichever way of measuring the longtitude of perihelion we choose. The relations between ω, ε, and ϖ, ϵ, are these,

$$\left.\begin{aligned} d\omega &= d\varpi - \cos i\, d\Omega \\ d\varepsilon &= d\epsilon - \cos i\, d\Omega \end{aligned}\right\}.$$

As for $\frac{da}{dt}$ and $\frac{de}{dt}$ they are the same whether we use ϖ and ϵ or ω and ε, since $\frac{dR}{d\varpi} = \frac{dR}{d\omega}$, and $\frac{dR}{d\epsilon} = \frac{dR}{d\varepsilon}$.

If we take the epoch c, $\left(\frac{dR}{d\epsilon} + \frac{dR}{d\varpi}\right)$ in the expression for $\frac{de}{dt}$ must be replaced by $\frac{dR}{d\varpi}$ or $\frac{dR}{d\omega}$.

In combining the above formulæ with the expressions given in Airy's *Tracts* or Pratt's *Mechanics*, the sign of R must be changed, as I have followed Lagrange in taking the disturbing function positive, when the disturbing force tends to increase the coordinates.

* The term containing t as a factor should be omitted in $\frac{d\epsilon}{dt}$, as it is introduced by the variation of n. See Lagrange, *Mem. Inst.* 1808, p. 64.

Throughout, H may be replaced by the equivalent expressions $na^2\sqrt{(1-e^2)}$ or $\sqrt{\{\mu a(1-e^2)\}}$. The complete series of the variations in the different cases is subjoined, for facility of reference:

$$\frac{da}{dt}=\frac{2}{na}\frac{dR}{d\epsilon},$$

$$\frac{de}{dt}=\frac{1-e^2}{na^2e}\cdot\frac{dR}{d\epsilon}-\frac{\sqrt{(1-e^2)}}{na^2e}\left(\frac{dR}{d\epsilon}+\frac{dR}{d\varpi}\right),$$

$$\frac{d\varpi}{dt}=\frac{\sqrt{(1-e^2)}}{na^2e}\cdot\frac{dR}{de},$$

$$\frac{d\epsilon}{dt}=-\frac{2}{na}\frac{dR}{da}+\frac{1}{na^2e}\left\{\sqrt{(1-e^2)}-(1-e^2)\right\}\frac{dR}{de},$$

$$\frac{d\Omega}{dt}=\frac{1}{na^2\sqrt{(1-e^2)}\,.\sin i}\frac{dR}{di},$$

$$\frac{di}{dt}=-\frac{1}{na^2\sqrt{(1-e^2)}\,.\sin i}\frac{dR}{d\Omega},$$

when the angles are measured from a fixed line in the plane. When they are measured from the line of nodes, the only change in the expressions for $\frac{da}{dt}$ and $\frac{de}{dt}$ is to put ε and ω for ϵ and ϖ. Instead of $\frac{d\epsilon}{dt}$ and $\frac{d\varpi}{dt}$, we have

$$\frac{d\omega}{dt}=\frac{\sqrt{(1-e^2)}}{na^2e}\frac{dR}{de}-\frac{\cos i}{na^2\sqrt{(1-e^2)}\,.\sin i}\frac{dR}{di},$$

$$\frac{d\varepsilon}{dt}=-\frac{2}{na}\frac{dR}{da}$$

$$+\frac{1}{na^2e}\left\{\sqrt{(1-e^2)}-(1-e^2)\right\}\frac{dR}{de}-\frac{\cos i}{na^2\sqrt{(1-e^2)}\sin i}\frac{dR}{di},$$

and the expression for $\frac{di}{dt}$ becomes

$$=-\frac{1}{na^2\sqrt{(1-e^2)}\sin i}\frac{dR}{d\Omega}+\frac{\cos i}{na^2\sqrt{(1-e^2)}\sin i}\left(\frac{dR}{d\omega}+\frac{dR}{d\varepsilon}\right).$$

The change in this expression, it must be remembered, arises, not from any change in the *value* of $\frac{di}{dt}$, but from the different manner in which Ω is involved in R.

When we use the epoch c instead of ϵ or ε in all the above expressions, $\frac{dR}{d\epsilon}$ or $\frac{dR}{d\varepsilon}$ is replaced by $\frac{dR}{dc}$, and $\frac{dR}{d\varpi}+\frac{dR}{d\epsilon}$ or $\frac{dR}{d\omega}+\frac{dR}{d\varepsilon}$ by $\frac{dR}{d\varpi}$ or $\frac{dR}{d\omega}$; and instead of $\frac{d\epsilon}{dt}$ or $\frac{d\varepsilon}{dt}$ we have

$$\frac{dc}{dt} = -\frac{2}{na}\frac{dR}{da} - \frac{1-e^2}{na^2e}\frac{dR}{de}.$$

ON SYMBOLICAL GEOMETRY.

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INTRODUCTORY REMARKS.

The present paper is an attempt towards constructing a symbolical geometry, analogous in several important respects to what is known as symbolical algebra, but not identical therewith; since it starts from other suggestions, and employs, in many cases, other rules of combination of symbols. One object aimed at by the writer has been (he confesses) to illustrate, and to exhibit under a new point of view, his own theory, which has in part been elsewhere published, of algebraic quaternions. Another object, which interests even him much more, and will probably be regarded by the readers of this Journal as being much less unimportant, has been to furnish some new materials towards judging of the general applicability and usefulness of some of those principles respecting symbolical language which have been put forward in modern times. In connexion with this latter object he would gladly receive from his readers some indulgence, while offering the few following remarks.

An opinion has been formerly published* by the writer of the present paper, that it is possible to regard Algebra as a *science*, (or more precisely speaking) as a *contemplation*, in some degree *analogous to Geometry*, although not to be confounded therewith; and to separate it, as such, in our conception, from its own *rules of art* and *systems of expression:* and that when so regarded, and so separated, its ultimate subject-matter is found in what a great metaphy-

* *Trans. Royal Irish Acad.*, vol. XVII. Dublin, 1835.

sician has called the inner intuition of *time.* On which account, the writer ventured to characterise Algebra as being the *Science of Pure Time;* a phrase which he also expanded into this other: that it is (ultimately) the Science of *Order and Progression.* Without having as yet seen cause to abandon that former view, however obscurely expressed and imperfectly developed it may have been, he hopes that he has since profited by a study, frequently resumed, of some of the works of Professor Ohm, Dr. Peacock, Mr. Gregory, and some other authors; and imagines that he has come to seize their meaning, and appreciate their value, more fully than he was prepared to do, at the date of that former publication of his own to which he has referred. The whole theory of the laws and logic of symbols is indeed one of no small subtlety; insomuch that (as is well known to the readers of the *Cambridge Mathematical Journal,* in which periodical many papers of great interest and importance on this very subject have appeared) it requires a close and long-continued attention, in order to be able to form a judgment of any value respecting it: nor does the present writer venture to regard his own opinions on this head as being by any means sufficiently matured; much less does he desire to provoke a controversy with any of those who may perceive that he has not yet been able to adopt, in all respects, their views. That he has adopted *some* of the views of the authors above referred to, though in a way which does not seem to himself to be contradictory to the results of his former reflexions; and especially that he feels himself to be under important obligations to the works of Dr. Peacock upon Symbolical Algebra, are things which he desires to record, or mark, in some degree, by the very *title* of the present communication; in the course of which there will occur opportunities for acknowledging part of what he owes to other works, particularly to Mr. Warren's Treatise on the Geometrical Representation of the Square Roots of Negative Quantities.

Observatory of Trinity College, Dublin, Oct. 16, 1845.

Uniliteral and Biliteral Symbols.

1. In the following pages of an attempt towards constructing a symbolical geometry, it is proposed to employ (as usual) the roman capital letters A, B, &c., with or without accents, as symbols of *points* in space; and to make use (at first) of binary combinations of those letters, as symbols of straight *lines:* the symbol of the beginning

of the line being written (for the sake of some analogies*) towards the right hand, and the symbol of the end towards the left. Thus BA will denote the line *to* B *from* A; and is not to be confounded with the symbol AB, which denotes a line having indeed the same extremities, but drawn in the opposite direction. A biliteral symbol, of which the two component letters denote determined and different points, will thus denote a finite straight line, having a determined length, direction, and situation in space. But a biliteral symbol of the particular form AA may be said to be a *null* line, regarded as the limit to which a line tends, when its extremities tend to coincide: the conception or at least the name and symbol of such a line being required for symbolic generality. All lines BA which are not null, may be called by contrast *actual;* and the two lines AB and BA may be said to be the *opposites* of each other. It will then follow that a null line is its own opposite, but that the opposites of two actual lines are always to be distinguished from each other.

On the mark =.

2. An *equation* such as $B = A$ ....(1),

between two uniliteral symbols, may be interpreted as denoting that A and B are *two names for one common point;* or that a point B, determined by one geometrical process, coincides with a point A determined by another process. When a formula of the kind (1) holds good, in any calculation, it is allowed to *substitute*, in any other part of that calculation, either of the two equated symbols for the other; and every other equation between two symbols of one common class must be interpreted so as to allow a similar substitution. We shall not violate this principle of symbolical language by interpreting, as we shall interpret, an equation such as

$$DC = BA \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

between two biliteral symbols, as denoting that the two lines,†

* The writer regards the line to B from A as being in some sense an interpretation or construction of the symbol $B - A$; and the evident possibility of reaching the point B, by going along that line from the point A, may, as he thinks, be symbolized by the formula $B - A + A = B$.

† The writer regards the relation between two lines, mentioned in the text, as a sort of interpretation of the following symbolic equation, $D - C = B - A$; which may also denote that the point D is ordinarily related (in space) to the point C as B is to A, and may in that view be also expressed by writing the *ordinal analogy*, D..C :: B..A; which admits of *inversion* and *alternation.* The same relation between four points may, as he thinks, be thus symbolically expressed, $D = B - A + C$. But by writing it as an equation between lines, he deviates less from received notation.

of which the symbols are equated, have *equal lengths and similar directions,* though they may have different situations in space: for if we call such lines *symbolically equal*, it will be allowed, in *this* sense of equality, which has indeed been already proposed by Mr. Warren, Dr. Peacock, and probably by some of the foreign writers referred to in Dr. Peacock's Report, as well as in that narrower sense which relates to magnitudes only, and for lines in space as well as for those which are in one plane, to assert that lines *equal* to the same line are equal to each other. (Compare *Euclid*, XI. 9.) It will also be true, that

$$D = B, \quad \text{if } DA = BA \text{ .... } (3),$$

or in words, that the ends of two symbolically equal lines coincide if the beginnings do so; a consequence which it is very desirable and almost necessary that we should be able to draw, for the purposes of symbolical geometry, but which would not have followed, if an equation of the form (2) had been interpreted so as to denote *only* equality of lengths, or *only* similarity of directions. The opposites of equal lines are equal in the sense above explained; therefore the equation (2) gives also this *inverse* equation,

$$CD = AB \text{ .... } (4).$$

Lines joining the similar extremities of symbolically equal lines are themselves symbolically equal (*Euc.* I. 33); therefore the equation (2) gives also this *alternate* equation,

$$DB = CA \text{ .... } (5).$$

The *identity* BA = BA gives, as its alternate equation,

$$BB = AA \text{ .... } (6),$$

which symbolic result may be expressed in words by saying that any two null lines are to be regarded as equal to each other. Lines equal to opposite lines may be said to be themselves opposite lines.

On the mark +.

3. The equation* $CB + BA = CA$ .... (7)

is true in the most elementary sense of the notation, when B is any point upon the finite straight line CA; but we propose now to *remove this restriction for the purposes of symbolical*

* On the plan mentioned in former notes, this equation would be written as follows:

$$(C - B) + (B - A) = C - A.$$

It might also be thus expressed: the ordinal relation of the point C to the point A is compounded of the relations of C to B and of B to A.

geometry, and to regard the formula (7) as being universally *valid, by definition, whatever three points of space may be denoted* by the three letters ABC. The equation (7) will then *express nothing about those points,* but will serve to *fix the interpretation of the mark + when inserted between any two symbols of lines;* for if we meet any symbol formed by such insertion, suppose the symbol HG + FE, we have only to draw, or conceive drawn, from any assumed point A, a line BA = FE, and from the end B of the line so drawn, a new line CB = HG; and then the proposed symbol HG + FE will be interpreted by (7) as denoting the line CA, or at least a line equal thereto. In like manner, by defining that

$$DC + CB + BA = DA \; \ldots\ldots\ldots\ldots \; (8),$$

we shall be able to interpret any symbol of the form

$$KI + HG + FE,$$

as denoting a determined (actual or null) line; at least if we now regard a line as *determined* when it is *equal* to a determined line: and similarly for any number of biliteral symbols, connected by marks + interposed. Calling *this* act of connection of symbols, the operation of *addition;* the added symbols, *summands;* and the resulting symbol, a *sum;* we may therefore now say, that the sum of any number of symbols of given lines is itself a symbol of a determined line; and that this symbolic sum of lines represents the *total* (or final) *effect* of all those successive rectilinear *motions,* or translations of a point in space, which are represented by the several summands. This *interpretation of a symbolic sum of lines* agrees with the conclusions already published by the authors above alluded to; though the modes of symbolically obtaining and expressing it, here given, may possibly be found to be new. The same interpretation satisfies, as it ought to do, the condition that the sums of equals shall be equal (compare the demonstration of *Euclid,* XI. 10); and also this other condition, almost as much required for the advantageous employment of symbolical language, that those lines which, when added to equal lines, give equal sums, shall be themselves equal lines: or that

$$FE = DC, \quad \text{if } FE + BA = DC + BA \; \ldots\ldots \; (9).$$

It shews too that the sum of two opposite lines, and generally that the sum of all the successive sides of any closed polygon, or of lines respectively equal to those sides, is a null line: thus

$$AA = AB + BA = AC + CB + BA = \&c. \; \ldots\ldots \; (10).$$

The symbolic sum of any two lines is found to be *independent of their order*, in virtue of the same interpretation; so that the equation

$$FE + HG = HG + FE \ldots\ldots\ldots\ldots (11),$$

is true, in the present system, *not as an independent definition*, but rather as one of the modes of *symbolically expressing that elementary theory of geometry*, (*Euclid*, I. 33), on which was founded the rule for deducing, from any equation (2) between lines, the *alternate* equation (5). For if we assume, as we may, that three points A, B, C, have been so chosen as to satisfy the equations FE = BA, HG = CA; and that a fourth point D is chosen so as to satisfy the equation DC = BA; the same points will then, by the theorem just referred to, satisfy also the equation DB = CA; and the truth of the formula (11) will be proved, by observing that each of the two symbols which are equated in that formula is equal to the symbol DA, in virtue of the definition (7) of +, without any new definition: since

$$FE + HG = DC + CA = DA = DB + BA = HG + FE.$$

A like result is easily shown to hold good, for any number of summands; thus

$$FE + HG + KI = KI + HG + FE \ldots\ldots (12);$$

since the first member of this last equation may be put successively under the forms

$$(FE + HG) + KI,\ KI + (FE + HG),\ KI + (HG + FE),$$

and finally under the form of the second member; the stages of this successive transformation of symbols admitting easily of geometrical interpretations: and similarly in other cases. *Addition of lines in space* is therefore generally (as Mr. Warren has shewn it to be for lines in a single plane) a *commutative operation;* in the sense that the summands may interchange their places, without the sum being changed. It is also an *associative* operation, in the sense that any number of successive summands may be associated into one group, and collected into one partial sum (denoted by enclosing these summands in parentheses); and that then this partial sum may be added, as a single summand, to the rest: thus

$$(KI + HG) + FE = KI + (HG + FE) = KI + HG + FE \ldots (13).$$

On the mark –.

4. The equation* $CA - BA = CB \ldots\ldots\ldots\ldots (14)$

* On the plan mentioned in some former notes, this equation would take the form

$$(C - A) - (B - A) = C - B.$$

is true, in the most elementary sense of the notation, when B is on CA; but we may remove this restriction by a *definitional extension* of the formula (14), for the purposes of symbolical geometry, as has been done in the foregoing article with respect to the formula (7); and then the equation (14), so extended, will express *nothing about the points* A, B, C, but will serve to fix the *interpretation of any symbol*, such as KI - FE, formed by *inserting the mark - between the symbols of any two lines*. This general meaning of the effect of the mark -, so inserted, is consistent with the particular interpretation which suggested the formula (14); it is also consistent with the usual symbolical opposition between the effects of + and -; since the comparison of (14) with (7) gives the equations

$$(\mathrm{CA} - \mathrm{BA}) + \mathrm{BA} = \mathrm{CA} \quad \ldots\ldots\ldots (15),$$

and

$$(\mathrm{CB} + \mathrm{BA}) - \mathrm{BA} = \mathrm{CB} \quad \ldots\ldots\ldots (16),$$

either of which two equations, if regarded as a general formula, and combined with the formula (7), would include, reciprocally, the definition (14) of -, and might be substituted for it.

Symbolical *subtraction* of one line from another is thus equivalent to the *decomposition* of a given rectilinear *motion* (CA) into two others, of which one (BA) is given; or to the *addition of the opposite* (AB) of the line which was to be subtracted: so that we may write the symbolical equation

$$- \mathrm{BA} = + \mathrm{AB} \quad \ldots\ldots\ldots\ldots (17),$$

because the second member of (14) may be changed by (7) to CA + AB. These conclusions respecting symbolical subtraction of lines, differ only in their notation, and in the manner of arriving at them, from the results of the authors already referred to, so far as the present writer is acquainted with them. In the present notation, when an isolated biliteral symbol is preceded by + or -, we may still interpret it as denoting a line, if we agree to prefix to it, for the purpose of such interpretation, the symbol of a null line; thus we may write

$$+ \mathrm{AB} = \mathrm{AA} + \mathrm{AB} = \mathrm{AB}, \quad - \mathrm{AB} = \mathrm{BB} - \mathrm{AB} = \mathrm{BA} \ldots (18);$$

+ AB will, therefore, on this plan, be another symbol for the line AB itself, and - AB will be a symbol for the opposite line BA.

Abridged Symbols for Lines.

5. Some of the foregoing formulæ may be presented more concisely, and also in a way more resembling ordinary Algebra, by using now some new *uniliteral* symbols, such

as the small roman letters a, b, &c., with or without accents, as symbols of lines, instead of binary combinations of the roman capitals, in cases where the lines which are compared are not supposed to have necessarily any common point, and generally when the *situations* of lines are disregarded, but not their lengths nor their directions. Thus we shall have, instead of (11) and (12), (13), (15) and (16), these other formulæ of the present Symbolical Geometry, which agree in all respect with those used in Symbolical Algebra:

$$\mathrm{a} + \mathrm{b} = \mathrm{b} + \mathrm{a}, \quad \mathrm{a} + \mathrm{b} + \mathrm{c} = \mathrm{c} + \mathrm{b} + \mathrm{a} \ldots\ldots (19);$$

$$(\mathrm{c} + \mathrm{b}) + \mathrm{a} = \mathrm{c} + (\mathrm{b} + \mathrm{a}) = \mathrm{c} + \mathrm{b} + \mathrm{a} \ \ldots\ldots (20);$$

$$(\mathrm{b} - \mathrm{a}) + \mathrm{a} = \mathrm{b}, \quad (\mathrm{b} + \mathrm{a}) - \mathrm{a} = \mathrm{b} \ldots\ldots\ldots\ldots (21);$$

and because the isolated but *affected* symbols + a, − a, may denote, by (18), the line a itself, and the opposite of that line, we have also here the usual *rule of the signs*,

$$+(+\mathrm{a}) = -(-\mathrm{a}) = +\mathrm{a}, \quad +(-\mathrm{a}) = -(+\mathrm{a}) = -\mathrm{a} \ldots (22).$$

Introduction of the marks × and ÷.

6. Continuing to denote lines by letters, the formula

$$(\mathrm{b} \div \mathrm{a}) \times \mathrm{a} = \mathrm{b} \ldots\ldots\ldots\ldots\ldots\ldots (23),$$

which is, for the relation between multiplication and division, what the first of the two formulæ (21) is for the relation between addition and subtraction, will be true, in the most elementary sense of the multiplication of a length by a number, for the case when the line b is the sum of several summands, each equal to the line a, and when the number of those summands is denoted by the quotient b ÷ a. And we shall now, for the purposes of symbolical generality, *extend* this formula (23), so as to make it be valid, *by definition, whatever two lines* may be denoted by a and b. The formula will then *express nothing respecting those lines* themselves, which can serve to distinguish them from any other lines in space; but will furnish a *symbolic condition,* which we must satisfy by the *general interpretation* of a *geometrical quotient,* and of the *operation of multiplying a line* by such a quotient.

To make such general interpretation consistent with the particular case where a quotient becomes a *quotity,* we are led to write

$$\mathrm{a} \div \mathrm{a} = 1, \quad (\mathrm{a} + \mathrm{a}) \div \mathrm{a} = 2, \quad \&\mathrm{c.} \ldots\ldots (24),$$

and conversely

$$1 \times \mathrm{a} = \mathrm{a}, \quad 2 \times \mathrm{a} = \mathrm{a} + \mathrm{a}, \quad \&\mathrm{c.} \ldots\ldots\ldots (25);$$

and because, when quotients can be thus interpreted as quotities, the four equations

$$(\mathrm{c} \div \mathrm{a}) + (\mathrm{b} \div \mathrm{a}) = (\mathrm{c} + \mathrm{b}) \div \mathrm{a} \ldots\ldots\ldots (26),$$

$$(\mathrm{c} \div \mathrm{a}) - (\mathrm{b} \div \mathrm{a}) = (\mathrm{c} - \mathrm{b}) \div \mathrm{a} \ldots\ldots\ldots (27),$$

$$(\mathrm{c} \div \mathrm{a}) \times (\mathrm{a} \div \mathrm{b}) = \mathrm{c} \div \mathrm{b} \ldots\ldots\ldots\ldots (28),$$

$$(\mathrm{c} \div \mathrm{a}) \div (\mathrm{b} \div \mathrm{a}) = \mathrm{c} \div \mathrm{b} \ldots\ldots\ldots\ldots (29),$$

are true in the most elementary sense of arithmetical operations on whole numbers, we shall now *define* that these four equations are valid, *whatever three lines* may be denoted by a, b, c; and thus shall have conditions for the general *interpretations of the four operations* $+ \; - \; \times \; \div$ *performed on geometrical quotients.*

We shall in this way be led to interpret a quotient of which the divisor is an actual line, but the dividend a null one, as being equivalent to the symbol $1 - 1$, or *zero*; so that

$$(\mathrm{a} - \mathrm{a}) \div \mathrm{a} = 0, \quad 0 \times \mathrm{a} = \mathrm{a} - \mathrm{a} \ldots\ldots (30).$$

Negative numbers will present themselves in the consideration of such quotients and products as

$$(-\mathrm{a}) \div \mathrm{a} = 0 - 1 = -1, \quad (-1) \times \mathrm{a} = -\mathrm{a}, \;\&\text{c.} \ldots (31);$$

fractional numbers in such formulæ as

$$\mathrm{a} \div (\mathrm{a} + \mathrm{a}) = 1 \div 2 = \tfrac{1}{2}, \quad \tfrac{1}{2} \times (\mathrm{a} + \mathrm{a}) = \mathrm{a}, \;\&\text{c.} \ldots (32);$$

and *incommensurable* numbers, by the conception of the connected *limits* of quotients and products, and by the formula, which symbolical language leads us to assume,

$$\left(\lim \frac{n}{m}\right) \times \mathrm{a} = \lim \left(\frac{n}{m} \times \mathrm{a}\right) \ldots\ldots\ldots (33).$$

If then we give the name of SCALARS to all numbers of the kind called usually *real*, because they are all contained on the one *scale* of progression of number from negative to positive infinity; and if we agree, for the present, to denote such numbers generally by small italic letters, a, b, c, &c.; and to insert the mark $\|$ between the symbols of two lines when we wish to express that the directions of those lines are either exactly similar or exactly opposite to each other, in each of which two cases the lines may be said to be *symbolically parallel;* we shall have generally two equations of the forms

$$\mathrm{b} \div \mathrm{a} = a, \; a \times \mathrm{a} = \mathrm{b}, \text{ when } \mathrm{b} \parallel \mathrm{a} \ldots\ldots (34).$$

That is to say, the *quotient of two parallel lines* is generally a *scalar number;* and, conversely, to multiply a given line (a) by a given scalar (or real) number a, is to determine a new

line (b) parallel to the given line (a), the direction of the one being similar or opposite to that of the other, according as the number is positive or negative, while the length of the new line bears to the length of the given line a ratio which is marked by the same given number. So that if A_0 A_1 A_a denote any three points on one common axis of rectilinear progression, which are related to each other, upon that axis, as to their order and their intervals, in the same manner as the three scalar numbers 0, 1, a, regarded as ordinals, are related to each other on the scale of numerical progression from $-\infty$ to $+\infty$, then the equations

$$A_aA_0 \div A_1A_0 = a, \quad a \times A_1A_0 = A_aA_0, \ldots\ldots (35)$$

will be true by the foregoing interpretations.

It is easy to see that this mode of interpreting a quotient of parallel lines renders the formulæ (26) (27) (28) (29) consistent with the received rules for performing the operations $+ - \times \div$ on what are called real numbers, whether they be positive or negative, and whether commensurable or incommensurable; or rather reproduces those rules as consequences of those formulæ.

On Vectors, and Geometrical Quotients in general.

7. The other chief relation of directions of lines in space, besides parallelism, is perpendicularity; which it is not unusual to denote by writing the mark $\perp$ between the symbols of two perpendicular lines. And the other chief class of geometrical quotients which it is important to study, as preparatory to a general theory of such quotients, is the class in which the dividend is a line perpendicular to the divisor. A quotient of this latter class we shall call a VECTOR, to mark its connection (which is closer than that of a *scalar*) with the conception of *space*, and for other reasons which will afterwards appear: and if we agree to denote, for the present, such vector quotients (of perpendicular lines) by small Greek letters, in contrast to the scalar class of quotients (of parallel lines) which we have proposed to denote by small italic letters, we shall then have generally two equations of the forms

$$c \div a = \alpha, \quad c = \alpha \times a, \quad \text{if } c \perp a \ldots\ldots\ldots (36).$$

Any line e may be put under the form c + b, in which b ∥ a, and c $\perp$ a; a *general geometrical quotient* may therefore, by (26) (34) (36), be considered as the ***symbolic sum of a scalar and a vector***, zero being regarded as a common limit of quotients of these two classes; and consequently, if we

adopt the notation just now mentioned, we have generally an equation of the form

$$e \div a = \alpha + a \ldots\ldots\ldots\ldots\ldots (37).$$

This *separation of the scalar and vector parts* of a general geometrical quotient corresponds (as we see) to the decomposition, by *two separate projections*, of the dividend line into two other lines of which it is the symbolic sum, and of which one is parallel to the divisor line, while the other is perpendicular thereto. To be able to mark on some occasions more distinctly, in writing, than by the use of two different alphabets, the conception of such separation, we shall here introduce two new symbols of operation, namely the abridged words Scal. and Vect., which, where no confusion seems likely to arise from such farther abridgment, we shall also denote more shortly still by the letters S and V, prefixing them to the symbol of a general geometrical quotient in order to form separate symbols of its scalar and vector parts: so that we shall now write generally, for any two lines a and e,

$$e \div a = \text{Vect.}(e \div a) + \text{Scal.}(e \div a) \ldots\ldots (38);$$

or more concisely,

$$e \div a = \mathrm{V}(e \div a) + \mathrm{S}(e \div a) \ldots\ldots\ldots (39);$$

in which expression the order of the two summands may be changed, in virtue of the definition (26) of addition of geometrical quotients, because the order of the two partial dividends may be changed without preventing the dividend line e from being still their symbolic sum. A scalar cannot become equal to a vector, except by each becoming zero; for if the divisor of the vector quotient be multiplied separately by the scalar and the vector, the products of these two multiplications will be (by what has been already shown) respectively lines parallel and perpendicular to that divisor, and therefore not symbolically equal to each other, except it be at the limit where both become null lines, and are on that account regarded as equal. A scalar quotient $b \div a = \alpha$, $(b \parallel a)$, has been seen to denote the relative length and relative direction (as similar or opposite) of two parallel lines a, b: and in like manner a vector quotient $c \div a = a$, $(c \perp a)$, may be regarded as denoting the *relative length and relative direction* (depending on *plane* and *hand*) *of two perpendicular lines* a, c; or as indicating at once *in what ratio* the length of one line a must be altered (if at all) in order to become equal to the length of another line c, and also *round what axis*, perpendicular to both these two rectangular lines, the direction of the divisor line a must be caused or conceived to

turn, right-handedly, through a right angle, in order to attain the original direction of the dividend line c. A line drawn in the direction of this *axis of* (what is here regarded as) *positive rotation,* and having its length in the same ratio to some assumed *unit* of length as the length of the dividend to that of the divisor, may be called the INDEX of the vector. We shall thus be led to substitute, for any equation between two vector quotients, an equation between two lines, namely between their indices; for if we define that two vector quotients, such as $c \div a$ and $c' \div a'$ if $c \perp a$ and $c' \perp a'$, are *equal* when they have *equal indices,* we shall satisfy all conditions of symbolical equality, of the kinds already considered in connection with other definitions; we shall also be able to say that in every case of two such equal quotients, the two dividend lines (c and c') bear to their own divisor lines (a and a'), respectively, one common ratio of lengths, and one common relation of directions. We shall thus also, by (23), be able to *interpret the multiplication* of any given line a' by any given vector $c \div a$, *provided that the one is perpendicular to the index of the other,* as the operation of deducing from a' another line c', by altering (generally) its length in a given ratio, and by turning (always) its direction round a given axis of rotation, namely round the index of the vector, right-handedly, through a right angle. And we can now *interpret an equation between two general geometrical quotients,* such as

$$e' \div a' = e \div a. \ldots\ldots\ldots\ldots (40),$$

as being equivalent to a *system of two separate equations,* one between the scalar and another between the vector parts, namely the two following:

$$S(e' \div a') = S(e \div a);\ V(e' \div a') = V(e \div a). . (41);$$

of which each separately is to be interpreted on the principles already laid down; and which are easily seen (by considerations of similar triangles) to imply, when taken jointly, that the length of e' is to that of a' in the same ratio as the length of e to that of a; and also that the same rotation, round the index of either of the two equal vectors, which would cause the direction of a to attain the original direction of e, would also bring the direction of a' into that originally occupied by e'. At the same time we see how to interpret the operation of multiplying any given line a' by any given geometrical quotient $e \div a$ of two other lines, *whenever the three given lines* a, e, a', *are parallel to one common plane;* namely as being the complex operation of altering (generally) a given length in a given ratio, and of turning a given line

round a given axis, through a given amount of right-handed rotation, in order to obtain a certain new line e′, which may be thus denoted, in conformity with the definition (23),

$$e' = (e \div a) \times a' \ldots\ldots\ldots\ldots (42).$$

The relation between the four lines a, e, a′, e′, may also be called a *symbolic analogy*, and may be thus denoted:

$$e' : a' :: e : a \ldots\ldots\ldots\ldots (43);$$

a′ and e being the *means*, and e′ and a the *extremes* of the analogy. An analogy or equation of this sort admits (as it is easy to prove) of *inversion* and *alternation*; thus (43) or (42) gives, *inversely*,

$$a' : e' :: a : e, \quad a' \div e' = a \div e \ldots\ldots\ldots (44),$$

and *alternately*,

$$e' : e :: a' : a, \quad e' \div e = a' \div a \ldots\ldots\ldots (45).$$

These results respecting analogies between *co-planal lines*, that is, between lines which are in or parallel to one common plane, agree with, and were suggested by, the results of Mr. Warren. But it will be necessary to introduce other principles, or at least to pursue farther the track already entered on, before we can arrive at an interpretation of a *fourth proportional to three lines which are not parallel to any common plane:* or can interpret the multiplication of a line by a quotient of two others, when it is not perpendicular to what has been lately called the index of the vector part of that quotient.

(*To be continued.*)

ON THE QUADRATURE OF SURFACES OF THE SECOND ORDER.

By John H. Jellett, M.A., Fellow and Tutor of Trinity College, Dublin.

The object of the present memoir is to furnish a complete discussion of a question hitherto but partially investigated, viz. the quadrature of the five principal surfaces of the second order. The writers who have hitherto treated of this subject have, as far as I am aware, confined their attention to the ellipsoid, partly perhaps as being the most familiar, and partly because being a finite surface it might seem to admit the most complete solution of the question: but, as we shall see, some highly interesting results are to be derived from an investigation of the superficial area of the paraboloids, inasmuch as they only, among surfaces of the second order (excepting of course surfaces of revolution), admit of an algebraic expression for their superficial area. Before entering

into the question with regard to surfaces of the second order, it may be well to make a few observations on the problem of quadrature with regard to surfaces generally, for the purpose of pointing out some important differences which exist between it and the rectification of curves, to which it is supposed to be analogous. In the problem of rectification it is immaterial, as far as the possibility of the solution is concerned, what origin or what coordinates we choose; and that because this problem being of a nature perfectly definite, the possibility of solving it cannot be changed but by a change in the curve itself, and not in the lines or angles by which we measure it. In fact, for each curve there is but one problem of rectification, the possibility of solving which depends solely on the nature of the curve itself. But the problem of quadrature is essentially different. Here, the object being to find an expression for a portion of a surface enclosed within one or more curves, the possibility of attaining it will evidently depend as much on the nature of these curves as upon that of the surface. Now let $Vdudv$ denote the element of a surface, and it is evident that $du\int Vdv$, or the expression once integrated, will denote an element finite in one direction, comprised within the four curves $v = \phi(u)$, $v = \psi(u)$, $u = c$, $u = c + dc$, of which the first two are arbitrary, but the second two depend absolutely on the nature of the coordinate u. Thus, for example, if the element be $Vdxdy$, it is plain that $dx \,.\int Vdy$ will denote an element two of whose bounding curves are sections perpendicular to the axis of x. Now it is perfectly possible that, although the surface itself may be one of those ordinarily said to be susceptible of quadrature, this element may, from the nature of the bounding curves $u = c$, $u = c + dc$, be wholly insusceptible of it. Hence we see both the importance to the solution of this problem of a proper selection of coordinates, and the impossibility of pronouncing that a given surface is insusceptible of quadrature, with the same certainty with which we pronounce a given curve to be insusceptible of rectification. Having premised so much with regard to the general problem, I propose to shew, with regard to surfaces of the second order, (1) That the quadrature of the ellipsoid depends on the rectification of the focal conics of the reciprocal surface. (2) That the quadrature of the hyperboloids may to a certain extent be effected by the same means. (3) That the quadrature of the paraboloids may be effected algebraically, *i.e.* that it is possible to divide the surface by curves, such that the space intercepted between any two admits of an algebraic expression. For this purpose I must premise the following Lemmas:

LEMMA 1. Let $x_1 y_1 z_1$, $x_2 y_2 z_2$, &c., $X_1 Y_1 Z_1$, $X_2 Y_2 Z_2$, &c., be the coordinates of the angular points of two polygons having the same number of sides, and let it be supposed that

$$\frac{x_1}{X_1} = \frac{x_2}{X_2} = \&c. = \frac{a}{A},$$

$$\frac{y_1}{Y_1} = \frac{y_2}{Y_2} = \&c. = \frac{b}{B},$$

$$\frac{z_1}{Z_1} = \frac{z_2}{Z_2} = \&c. = \frac{c}{C},$$

then the solid contents of the pyramids whose common vertex is at the origin of coordinates, and whose bases are these polygons, are to each other in the ratio $abc : ABC$. It is evidently sufficient to prove this for pyramids with triangular bases; and since * the solid contents of these are

$$\tfrac{1}{6}.\{x_1(y_2z_3 - y_3z_2) + x_2(y_3z_1 - y_1z_3) + x_3(y_1z_2 - y_2z_1)\},$$

and $\tfrac{1}{6}.\{X_1(Y_2Z_3 - Y_3Z_2) + \&c. \qquad \}$,

the proposition is evident.

LEMMA 2. If on the surface of one of two ellipsoids having the same centre and coincident axes, there be described any closed curve, and upon the other the *corresponding* curve, *i. e.* the locus of *corresponding* points,† the solid sectors which have their vertex at the common centre, and whose bases are these curves, are to each other in the ratio of the solid contents of the ellipsoids themselves.

Let the semiaxes of the ellipsoids be a, b, c, A, B, C, and take an indefinitely small element of one surface and the corresponding element of the other, and it appears from Lemma 1, that the sectors whose vertex is at the common centre and of which these elements are the oases, will be to each other in the ratio $abc : ABC$, i. e. as the whole ellipsoids. As therefore the proposition is true of these elements of the solid sectors, and as to each element of one sector corresponds an element of the other, it is evidently true of the whole sectors.

LEMMA 3. Given two reciprocal‡ ellipsoids; the central radius vector to any point on one is coincident with and

* Vide Monge. *Journal de l'Ecole Polytechnique*, tom. VIII. p. 92. (Or Gregory's *Solid Geometry*, p. 17.)

† Corresponding points are those whose coordinates are proportional to the parallel axes.

‡ It is perhaps unnecessary to observe that two ellipsoids are said to be reciprocal when their axes are coincident and reciprocally proportional.

reciprocally proportional to the perpendicular on the tangent plane at the corresponding point on the other.

For if r, p, be the radius vector and perpendicular on the tangent plane at one point, and R, P, at the other, xyz, XYZ the coordinates of these points respectively,

$$r^2 = x^2 + y^2 + z^2$$
$$= \frac{a^2}{A^2} X^2 + \frac{b^2}{B^2} Y^2 + \frac{c^2}{C^2} Z^2 = \frac{X^2}{A^4} + \frac{Y^2}{B^4} + \frac{Z^2}{C^4} = \frac{1}{P^2}.$$

Similarly $R = \frac{1}{p}$, and the cosines of the angles which r makes with the axes are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$, or $\frac{PX}{A^2}, \frac{PY}{B^2}, \frac{PZ}{C^2}$, which are the cosines of the angles made by P with the axes. Hence it appears that r is coincident with and reciprocally proportional to P.

Prop. I. If p be the perpendicular from the centre of an ellipsoid on the tangent plane, θ and ϕ the angles which determine its position, and a, b, c the semiaxes, the element of the surface is $\frac{a^2 b^2 c^2}{p^4} . \sin \theta \, d\theta \, d\phi$.*

For, let $d\alpha$ represent this element, and ds the solid sector of which it is the base, then $d\alpha = \frac{3ds}{p}$. Now let dS be the corresponding sector of the reciprocal surface, and (Lemma 2) $ds = \frac{abc}{ABC} . dS = a^2 b^2 c^2 \, dS$; and since

$$dS = \tfrac{1}{3} R^3 \sin \theta \, d\theta \, d\phi = \tfrac{1}{3} \frac{\sin \theta \, d\theta \, d\phi}{p^3},$$

$$d\alpha = \frac{a^2 b^2 c^2}{p^4} . \sin \theta \, d\theta \, d\phi. \quad \text{Q. E. D.}$$

Prop. II. To integrate the above expression for the ellipsoid.

Assume
$$m^2 = a^2 \sin^2 \theta + c^2 \cos^2 \theta,$$
$$n^2 = b^2 \sin^2 \theta + c^2 \cos^2 \theta;$$

then $p^2 = a^2 \sin^2 \theta \cos^2 \phi + b^2 \sin^2 \theta \sin^2 \phi + c^2 \cos^2 \theta$
$$= m^2 \cos^2 \phi + n^2 \sin^2 \phi,$$

* Since arriving at the above expression I found that it had been previously given, with a different demonstration, by Jacobi. He has, however, only applied it to the ellipsoid, and his memoir has very little in common with the present. The theorem in Prop. IX. of the present paper is, I believe, the first attempt to compare different portions of the ellipsoidal surface.

and $$d\alpha = a^2 b^2 c^2 \frac{\sin\theta\, d\theta\, d\phi}{(m^2\cos^2\phi + n^2\sin^2\phi)^2}:$$

or, if we assume $\tan w = \frac{n}{m} . \tan\phi$,

$$d\alpha = a^2 b^2 c^2 \sin\theta\, d\theta \left\{\frac{1}{mn^3}\sin^2 w + \frac{1}{m^3 n}\cos^2 w\right\}. dw.$$

The limits of ϕ and therefore of w, are o and 2π or o and $\frac{\pi}{2}$, provided that the result be multiplied by 4. Now

$$\int_0^{\frac{\pi}{2}} \cos^2 w\, dw = \int_0^{\frac{\pi}{2}} \sin^2 w dw = \frac{\pi}{4},$$

therefore $$\alpha = \pi a^2 b^2 c^2 \left\{\int \frac{\sin\ \theta\, d\theta}{m^3 n} + \int \frac{\sin\ \theta\, d\theta}{mn^3}\right\}.$$

Before proceeding further it may be well to shew the geometrical meaning of the first integration, or, in other words, to give a geometrical method of constructing the curves which bound the element denoted by the expression $\pi a^2 b^2 c^2 \sin\theta d\theta \left(\frac{1}{m^3 n} + \frac{1}{mn^3}\right)$. It is evident, from what has been said at the commencement of this paper, that the equations of these curves are $\theta = \text{const} = k$ and $\theta = k + dk$. Let x, y, z be the coordinates of a point on either of these curves; and since

$$\cos\theta = \frac{pz}{c^2} = \frac{1}{c^2} . \frac{z}{\sqrt{\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)}},$$

we shall have $$\frac{x^2}{a^4} + \frac{y^2}{b^4} - \frac{z^2 . \tan^2\theta}{c^4} = 0,$$

the equation of a cone whose intersection with the ellipsoid will give the required curve. Hence these curves may be constructed as follows. In any plane perpendicular to the axis of z describe a number of ellipses whose common centre is at the point where their plane cuts the axis, and whose axes are parallel to, and in the duplicate ratio of, those of a and b. The cones whose common vertex is at the centre of the ellipsoid, and whose bases are the ellipses so constructed, will cut the ellipsoid in the required curves.*

* [It may be readily shown that, when the three axes are unequal, the bounding curve never coincides with a line of curvature Legendre has expressed, by means of *products* of elliptic functions, the area of a portion of an ellipsoid

PROP. III. The portion of the superficial area of the ellipsoid included between any two of the curves described in the foregoing proposition, may be expressed by means of arcs of the focal conics of the reciprocal ellipsoid.

We have seen that if dS be the elementary area included between two consecutive curves of the foregoing species,

$$dS = \pi a^2 b^2 c^2 \sin\theta d\theta \left(\frac{1}{nm^3} + \frac{1}{n^3 m}\right).$$

Hence, if any two curves of this species be described, for one of which the value of θ is α, and for the other α', the value of the superficial area which they include is

$$S = \pi a^2 b^2 c^2 (I + I'), \text{ where}$$

$$I = \int_\alpha^{\alpha'} \frac{\sin\theta d\theta}{(b^2\sin^2\theta + c^2\cos^2\theta)^{\frac{3}{2}}(a^2\sin^2\theta + c^2\cos^2\theta)^{\frac{1}{2}}},$$

$$I' = \int_\alpha^{\alpha'} \frac{\sin\theta d\theta}{(a^2\sin^2\theta + c^2\cos^2\theta)^{\frac{3}{2}}(b^2\sin^2\theta + c^2\cos^2\theta)^{\frac{1}{2}}},$$

The first of these integrals denotes the arc of an ellipse, and the second that of a hyperbola, it being supposed that $a > b > c$.

Assume $a^2 - c^2 = a^2 e^2$, $b^2 - c^2 = b^2 e'^2$, $e' \cos\theta = \sin u$,

and $I = \dfrac{1}{e'ab^3} \cdot \displaystyle\int \frac{\sec^2 u\, du}{\sqrt{\left(1 - \frac{e^2}{e'^2} \cdot \sin^2 u\right)}}$, or putting $x = \tan u$,

$$I = \frac{1}{e'ab^3} \cdot \int \frac{dx \sqrt{(1+x^2)}}{\sqrt{\left\{1 - \left(\frac{e^2}{e'^2} - 1\right) \cdot x^2\right\}}},$$

the limits of x being $\dfrac{e' \cos\alpha}{\sqrt{(1 - e'^2 \cos^2\alpha)}}$, $\dfrac{e' \cos\alpha'}{\sqrt{(1 - e'^2\cos^2\alpha')}}$.

Now let σ be the arc of an ellipse whose semiaxes are A and B, and it is known that $d\sigma = \dfrac{dy \sqrt{\{B^4 + (A^2 - B^2) y^2\}}}{B\sqrt{(B^2 - y^2)}}$,

or if we put $A^2 - B^2 = \dfrac{B^2}{K^2}$, and $y = BKx$,

$$d\sigma = BKdx \sqrt{\left(\frac{1 + x^2}{1 - K^2 x^2}\right)}.$$

bounded by four lines of curvature; but he has not found that his expression may be reduced to a single definite integral, which Mr. Jellett's investigations shew to be possible. See Legendre, *Traité des Fonctions Elliptiques*, vol. I. p. 350. Also Catalan, *Sur la Transformation des Variables dans les Intégrales Multiples. Mémoires Couronnés par l'Académie de Bruxelles*, 1839-40.]

To make this coincide with the foregoing expression for I, assume $\frac{e^2}{e'^2} - 1 = K^2$, and we shall find $\frac{A^2}{B^2} = \frac{\frac{1}{c^2} - \frac{1}{a^2}}{\frac{1}{b^2} - \frac{1}{a^2}}$. The ellipse therefore by means of which I is represented is similar to the focal ellipse of the reciprocal surface; and since there is nothing to determine its absolute magnitude, it may be taken to be the focal ellipse itself. If then σ be the arc of this ellipse, whose ordinates measured parallel to the minor axis are $y = \frac{BKe' \cos \alpha}{\sqrt{(1 - e'^2 \cos^2 \alpha)}}$, and $y' = \frac{BKe' \cos \alpha'}{\sqrt{(1 - e'^2 \cos^2 \alpha')}}$, we shall have $I = \frac{\sigma}{BKe'ab^3} = \frac{\sigma}{ab^3 B \sqrt{(e^2 - e'^2)}}$.

Now let a', b', c', be the semiaxes of the reciprocal ellipsoid, and it is evident that $e^2 - e'^2 = \frac{b'^2 - a'^2}{c'^2} = \frac{B^2}{C'^2}$,

therefore $$I = \frac{c'\sigma}{ab^3 B^2}.$$

By using a reduction precisely similar, we find $I' = \frac{c' \sigma'}{a^3 b B^2}$, where σ' is an arc of the focal hyperbola, whose extreme ordinates are $y = \frac{BK'e \cos \alpha}{\sqrt{(1 - e^2) \cos^2 \alpha}}$ and $y' = \frac{BK'e \cos \alpha'}{\sqrt{(1 - e^2 \cos^2 \alpha')}}$. Hence we find ultimately for the required superficial area the expression
$$S = \frac{\pi c^2 c'}{B^2} \cdot \left(\frac{a}{b}\ \sigma + \frac{b}{a}\ \sigma'\right),$$
or if we assume the reciprocal ellipsoid such that
$$aa' = bb' = cc' = B^2,$$
$$S = \pi c \cdot \left(\frac{a}{b} \cdot \sigma + \frac{b}{a} \cdot \sigma'\right).$$

PROP. IV. To construct geometrically the limits of the integrals I and I'.

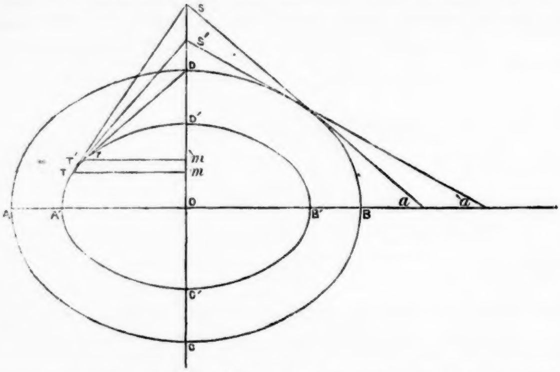
As the expressions for the limiting values of the ordinates are precisely similar for the two integrals, it will be sufficient to consider one of them.

We have seen that if y be one of the limiting ordinates, $y = \frac{BKe' \cos \alpha}{\sqrt{(1 - e'^2 \cos^2 \alpha)}}$, or, substituting for e' and K,
$$y = \frac{B^2 \cos \alpha}{\sqrt{(c'^2 \sin^2 \alpha + b'^2 \cos^2 \alpha)}} = \frac{B^2}{p \sec \alpha},$$

where p is the perpendicular drawn from the centre of the ellipse whose axes are c', b', (*i.e.* the principal section of the reciprocal ellipsoid which contains its greatest and mean axis) on the tangent, which makes the angle a with the axis c'.

Similarly $$y' = \frac{B^2}{p' . \sec a'}.$$

Hence we derive the following construction. Let $ACBD$ be the section of the reciprocal ellipsoid containing the greatest



and mean axis, $A'C'B'D'$ the focal ellipse of the same surface. Draw two tangents to the outer ellipse, making with the axis AB the angles a, a', and from the points s, s', where they cut the other axis, draw tangents to the inner ellipse. TT' is the arc which has become denoted by σ. For

$$Om = \frac{OD'^2}{OS} = \frac{B'^2}{p \sec a}$$

equals one of the limiting ordinates. The other is of course Om'.

The same construction precisely applies to the focal hyperbola: if then t and t' be the points of contact of the similarly drawn tangents to it, the surface of the ellipsoidal belt will be

$$S = \pi c . \left(\frac{a}{b} \, TT' + \frac{b}{a} \, tt'\right).$$

For the semiellipsoid, $a = \frac{\pi}{2}$, $a' = 0$; therefore the point S will go to infinity, S' will coincide with D, and the arc TT' will become $A't$. Similar reductions hold for the hyperbolic arc tt'.

PROP. V. The superficial area of the hyperboloids may, to a certain extent, be expressed in the same manner.

In the hyperboloid of one sheet, if θ be measured from the imaginary axis, we shall find the preceding investigation strictly applicable for all values of θ between $\frac{\pi}{2}$ and $\tan^{-1}\frac{c}{b}$. For values less than this, the integrals I and I' become imaginary, which is explained by observing that the cones whose intersections with the surface give the bounding curves, will in this case fall partly outside the surface. In the hyperboloid of two sheets it is necessary to measure θ from the real axis, and in this I and I' continue real from

$$\theta = 0 \text{ to } \theta = \tan^{-1}\frac{c}{a};$$

after this they become imaginary. The explanation of this is the same as for the hyperboloid of one sheet.

PROP. VI. To find what expression is to be substituted for $a^2 b^2 c^2 \dfrac{\sin\theta\, d\theta\, d\phi}{p^4}$ in the case of either of the paraboloids.

Adopting the usual mode of deriving the properties of the paraboloid from those of the ellipsoid or hyperboloid, we shall put $a^2 = mc$, $b^2 = nc$, and then make c infinite. Performing these operations we shall find

$$\frac{a^2 b^2 c^2}{p^4} = \frac{mnc^4}{(mc\sin^2\theta\cos^2\phi + nc\sin^2\theta\sin^2\phi + c^2\cos^2\theta)^2}$$

$$= \frac{mn}{\left(\dfrac{m\sin^2\theta\cos^2\phi + n\sin^2\theta\sin^2\phi}{c} + \cos^2\theta\right)^2} = \frac{mn}{\cos^4\theta}.$$

Hence we have $dS = mn.\dfrac{\sin\theta d\theta\,.\,d\phi}{\cos^4\theta}$

PROP. VII. To integrate the above expression and construct the bounding curves.

$$S = mn\int_a^{a'}\frac{\sin\theta d\theta}{\cos^4\theta}\int_0^{2\pi} d\phi = \frac{2\pi mn}{3}(\sec^3 a' - \sec^3 a).$$

The equations of the bounding curves are as before, $\theta = a$, $\theta = a'$: and since the equation of the tangent plane to the paraboloid is

$$z - z' = \frac{x'}{m}(x - x') + \frac{y'}{n}(y - y'),$$

if x, y be two of the coordinates of a point on one of these curves (the axis of z being the axis of the paraboloid), we shall have $\frac{x^2}{m^2} + \frac{y^2}{n^2} = \tan^2 a$, and $\frac{x^2}{m^2} + \frac{y^2}{n^2} = \tan^2 a'$. Hence the curves may be constructed as follows. In any plane perpendicular to the axis of the paraboloid, describe two ellipses whose axes are in the planes of the principal sections of the paraboloid and proportional to their parameters, and on these ellipses as bases erect two cylinders whose generatrices are parallel to the axis of the paraboloid. These cylinders will cut the surface in the required curves.

Prop. viii. The paraboloidal belt intercepted between any two of the curves described in the foregoing proposition, is proportional to the difference between the radii of curvature of either of the principal sections at the points where they intersect the bounding curves.

It appears from the preceding proposition that

$$S = \frac{2\pi mn}{3} . (\sec^3 a' - \sec^3 a),$$

a, a' being the angles made with the axis by the normal to the surface at any point on the bounding curves. Let R be the radius of curvature, and N the normal to the principal section whose semi-parameter is m at the point where it intersects the first of the bounding curves; then, since N is also normal to the surface, $\sec^3 a = \frac{N^3}{m^3} = \frac{R}{m}$, $\left(\text{since } R = \frac{N^3}{m^2}\right)$. Similarly $\sec^3 a' = \frac{R'}{m}$; therefore $S = \frac{2\pi n}{3}(R' - R)$. Q. E. D.

It is evident that if ρ, ρ' be the similar radii of curvature for the other principal section we shall have $S = \frac{2\pi m}{3}(\rho' - \rho)$. It appears also that if, with the same parameters and with the same principal planes, there be constructed two paraboloids, one elliptic, the other hyperbolic; the cylinders described in Prop. vii. will intercept on them portions whose superficial areas are the same.

Prop. ix. Let three curves be described on the surface of an ellipsoid along the first of which the perpendicular to the tangent plane makes with the axis of z the constant angle γ, along the second β with the axis of y, and along the third a with the axis of x, and let these angles be connected by the

equations $\frac{\tan\alpha}{a}=\frac{\tan\beta}{b}=\frac{\tan\gamma}{c}$;* then if A_3, A_2, A_1 be the included portions of the ellipsoidal surface, we shall have

$$\frac{A_3-A_2}{a^2}+\frac{A_1-A_3}{b^2}+\frac{A_2-A_1}{c^2}=0.$$

It appears from Prop. 3, that

$$dA_3=\frac{\pi a^2b^2c^2.\sin\theta d\theta}{\sqrt{\{(a^2\sin^2\theta+c^2\cos^2\theta).(b^2\sin^2\theta+c^2\cos^2\theta\}}}.$$
$$\left\{\frac{1}{a^2\sin^2\theta+c^2\cos^2\theta}+\frac{1}{b^2\sin^2\theta+c^2\cos^2\theta}\right\}:$$

And in the same way, if we had supposed the angle θ to be measured from the axis of y, we should have had

$$dA_2=\frac{\pi a^2b^2c^2\sin\theta d\theta}{\sqrt{\{(a^2\sin^2\theta+b^2\cos^2\theta)(c^2\sin^2\theta+b^2\cos^2\theta)\}}}.$$
$$\left(\frac{1}{a^2\sin^2\theta+b^2\cos\theta}+\frac{1}{c^2\sin^2\theta+b^2\cos^2\theta}\right).$$

and by measuring θ from the axis of x we should have a similar value for dA_1. Now if in the values of dA_1, dA_2, dA_3, respectively, for $\tan\theta$ we substitute ax, bx, cx, it is evident that the limits of integration with regard to x will be the same for all; and it is easy to see that the values of dA_1, dA_2, dA_3, may be put under the following forms,

$$dA_1=\pi b^2c^2\{2+(b^2+c^2)x^2\}(1+a^2x^2)^2 dL....(1),$$
$$dA_2=\pi a^2c^2\{2+(a^2+c^2)x^2\}(1+b^2x^2)^2 dL....(2),$$
$$dA_3=\pi a^2b^2\{2+(a^2+b^2)x^2\}(1+c^2x^2)^2 dL....(3),$$

where $dL=\frac{xdx}{(1+a^2x^2)^{\frac{3}{2}}.(1+b^2x^2)^{\frac{3}{2}}(1+c^2x^2)^{\frac{3}{2}}}.$

Multiply equation (1) by $(b^2-c^2)a^2$, equation (2) by $(c^2-a^2)b^2$, and equation (3) by $(a^2-b^2)c^2$, and add them, and it is easy to see that the right-hand member of the new equation will vanish, hence

$$(b^2-c^2)a^2dA_1+(c^2-a^2).b^2dA_2+(a^2-b^2)c^2dA_3=0,$$

a result which may be put under the form

$$\frac{d(A_1-A_2)}{a^2}+\frac{d(A_1-A_3)}{b^2}+\frac{d(A_2-A_1)}{c^2}=0,$$

* A relation analogous to this subsists between the perpendiculars on the tangents at the extremities of the elliptical arcs used in Fagnani's theorem, for if α be the angle made with the axis of x by the perpendicular corresponding to the arc which terminates at the extremity of that axis (a) and β the similar angle for the axis b, we shall have $\frac{\tan\alpha}{a}=\frac{\tan\beta}{b}$, as is easily seen.

And since A_1, A_2, A_3, all begin together, the proposition is evident. If instead of supposing A_1, A_2, A_3, to be bounded, each by a single curve, we conceive each of these letters to denote the space included between two such curves, the same theorem holds, provided that the curves of the second series are connected by the same equations as those of the first.

ON THE POLAR EQUATION TO A CHORD OF A CONIC SECTION.

By the Rev. PERCIVAL FROST, M.A., St. John's College.

IN a previous number of the *Mathematical Journal* having noticed a form of the polar equation to the tangent to a conic section, I think that the corresponding equation to the chord, which appears nearly in the same form, may be thought worthy of notice by some of the readers of the Journal.

Let the equation to the conic section be

$$\frac{c}{r} = 1 + e \cos \theta,$$

$\alpha + \beta$, $\alpha - \beta$ the values of θ which correspond to the points of intersection of the chord and conic section, and

$$\frac{c}{r} = m \cos \theta + n \sin \theta$$

the equation to the chord.

At the points of intersection we obtain by equating the sides of the equations

$$(m - e) \cos (\alpha - \beta) + n \sin (\alpha - \beta) = 1,$$
$$(m - e) \cos (\alpha + \beta) + n \sin (\alpha + \beta) = 1.$$

Hence $\quad (m - e) \cos \alpha \cos \beta + n \sin \alpha \cos \beta = 1,$

and $\quad (m - e) \sin \alpha \sin \beta - n \cos \alpha \sin \beta = 0;$

then $\dfrac{m - e}{\cos \alpha} = \dfrac{n}{\sin \alpha} = \dfrac{(m - e) \cos \alpha + n \sin \alpha}{\cos^2 \alpha + \sin^2 \alpha} = \dfrac{\sec \beta}{1} = \sec \beta.$

Therefore the equation to the chord of the conic section is

$$\frac{c}{r} = (e + \sec \beta \cos \alpha) \cos \theta + \sec \beta \sin \alpha \sin \theta$$
$$= \sec \beta \cos (\theta - \alpha) + e \cos \theta.$$

COR. If $\beta = 0$, we obtain the equation to the tangent at the point $\theta = \alpha$,

$$\frac{c}{r} = \cos (\theta - \alpha) + e \cos \theta.$$

By means of this equation the problems proposed in vol. III. p. 87, may be readily solved. For, since the equation to the chord may be written

$$\frac{c \cos \beta}{r} = \cos (\theta - \alpha) + e \cos \beta \cos \theta,$$

this chord touches the conic section whose eccentricity and latus rectum are $e \cos \beta$ and $2c \cos \beta$, the point of contact being in the line bisecting the angle between the distance; if this angle be constant, the conic section is the envelope of the chords.

If the focal distance corresponding to the angle $\alpha - \beta$ be produced, α', β' the values of α, β, corresponding to the produced focal distance and the other

$$\alpha' - \beta' = \alpha + \beta,$$

$$\alpha' + \beta' = \alpha - \beta + \pi,$$

therefore $$2\beta' = \pi - 2\beta,$$

$$\beta' = \frac{\pi}{2} - \beta.$$

And the envelope to the corresponding chord has for its equation

$$\frac{c \sin \beta}{r} = \cos (\theta - \alpha') + e \sin \beta \cos \theta,$$

and $$c^2 = (c \sin \beta)^2 + (c \cos \beta)^2,$$

$$e^2 = (e \sin \beta)^2 + (e \cos \beta)^2,$$

which prove the propositions.

Several problems may be conveniently solved by means of this equation.

Prob. 1. If the angle between two focal distances be bisected by a third which remains fixed in position, the chords joining the extremities of the two focal distances, as they change their position, always pass through a fixed point whose locus is the directrix.

The equation to any chord is

$$\frac{c \cos \beta}{r} = \cos (\theta - \alpha) + e \cos \beta \cos \theta;$$

therefore at the point of intersection with any other, α being constant,

$$\cos (\theta - \alpha) = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

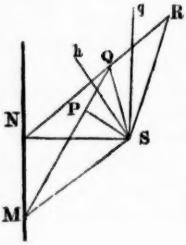
and $$\frac{c}{r} = e \cos \theta \ldots\ldots\ldots\ldots\ldots (2);$$

therefore, by (1), $$\theta = \alpha \mp \frac{\pi}{2} \ldots\ldots\ldots\ldots\ldots (3),$$

and by (2) the locus is the directrix (4). Hence the following geometrical construction may be easily performed.

Prob. 2. If three points P, Q, R be a conic section, and the focus be given, to construct the directrix.

Bisect the angles PSQ, QSR, by Sh and Sq. Draw SM and SN perpendicular to Sh, Sq. Produce QP and RQ to meet them in M and N. Join MN, which is the directrix, as appears from (3) and (4) of last problem.



Prob. 3. The locus of the intersection of chords drawn so that β in both is the same, and the difference between the α' constant, is a conic section.

ON THE REDUCTION OF $\dfrac{du}{\sqrt{U}}$, WHEN U IS A FUNCTION OF THE FOURTH ORDER.

By Arthur Cayley, M.A., Fellow of Trinity College.

It is well known that the transformation of this differential expression into a similar one, in which the function in the denominator contains only even powers of the corresponding variable, is the first step in the process of reducing $\int\dfrac{du}{\sqrt{U}}$ to elliptic integrals. And, accordingly, the different modes of effecting this have been examined, more or less, by most of those who have written on the subject. The simplest supposition, that adopted by Legendre, and likewise discussed in some detail by Guderman, is that (u) is a fraction, the numerator and denominator of which are linear functions of the new variable. But the theory of this transformation admits of being developed further than it has yet been done, as regards the equation which determines the modulus of the elliptic function. This may be effected most easily as follows.

Suppose

$$U = a + 4bu + 6cu^2 + 4du^3 + eu^4,$$

$$P = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Also let $P' = a'x'^4 + 4b'x'^3y' + 6c'x'^2y'^2 + 4d'x'y'^3 + e'y'^4$ be what P becomes after writing

$$x = \lambda x' + \mu y',$$

$$y = \lambda_{,} x' + \mu_{,} y':$$

and let $U' = a' + 4b'u' + 6c'u'^2 + 4d'u'^3 + e'u'^4$.

Suppose, moreover,

$$\begin{aligned} k &= \lambda\mu_{,} - \lambda_{,}\mu, \\ I &= ae - 4bd + 3c^2, \\ I' &= a'e' - 4b'd' + 3c'^2, \\ J &= ace - ad^2 - eb^2 - c^3 + 2bdc, \\ J' &= a'c'e' - a'd'^2 - e'b'^2 - c'^3 + 2b'd'c'; \end{aligned}$$

we have evidently

$$xdy - ydx = k.(x'dy' - y'dx'),$$

or

$$\frac{xdy - ydx}{P^{\frac{1}{2}}} = k.\frac{x'dy' - y'dx'}{P'^{\frac{1}{2}}}.$$

Or, writing

$$u = \frac{y}{x}, \quad u' = \frac{y'}{x'};$$

and therefore

$$\frac{xdy - ydx}{P^{\frac{1}{2}}} = \frac{du}{U^{\frac{1}{2}}}, \quad \frac{x'dy' - y'dx'}{P'^{\frac{1}{2}}} = \frac{du'}{U'^{\frac{1}{2}}}$$

$$\frac{du}{U^{\frac{1}{2}}} = k\,\frac{du'}{U'^{\frac{1}{2}}},$$

the equation between u and u' being

$$u = \frac{\lambda + \mu u'}{\lambda_{,} + \mu_{,}u'}.$$

Next, to determine the relations between the coefficients of U and U'. Since P, P' are obtained from each other by linear transformations (*Math. Journal*, vol. IV. p. 208), we have between the coefficients of these functions and of the transforming equations, the relations

$$I' = k^4.I,$$
$$J' = k^6.J;$$

whence also

$$\frac{J'^2}{I'^3} = \frac{J^2}{I^3}.$$

Suppose now $\quad U' = a'(1 + pu'^2)(1 + qu'^2)$,

or $\quad b' = 0, \quad d' = 0, \quad 6c' = a'(p + q), \quad e' = a'pq;$

whence also

$$I' = \frac{a'^2}{12}.(p^2 + q^2 + 14pq),$$

$$J' = \frac{a'^3}{216}(p + q).(34pq - p^2 - q^2);$$

$$p^2 + q^2 + 14pq = 12 \cdot \frac{k^4}{a'^2} I,$$

$$(p+q) \cdot (34pq - p^2 - q^2) = 216 \cdot \frac{k^6}{a'^3} J;$$

$$\therefore \frac{(p+q)^2 \cdot (34pq - p^2 - q^2)^2}{(p^2 + q^2 + 14pq)^3}$$

$$= \frac{27J^2}{I^3}, \quad \text{whence also } \frac{108pq\,(p-q)^4}{(p^2+q^2+14pq)^3} = 1 - \frac{27J^2}{I^3},$$

which determines the relation between p and q. Also

$$\frac{k}{\sqrt{a'}} = \left(\frac{p^2 + q^2 + 14pq}{12I}\right)^{\frac{1}{4}},$$

so that $\dfrac{du}{\sqrt{U}} = \left(\dfrac{p^2+q^2+14pq}{12I}\right)^{\frac{1}{4}} \dfrac{du'}{\{(1+pu'^2)(1+qu'^2)\}^{\frac{1}{2}}}$.

If in particular $p = -1$, writing also $-q$ for q,

$$\frac{du}{\sqrt{U}} = \left(\frac{q^2 + 14q + 1}{12I}\right)^{\frac{1}{4}} \frac{du'}{\{(1-u'^2)(1-qu'^2)\}^{\frac{1}{2}}},$$

where $\dfrac{108q(1-q)^4}{(q^2+14q+1)^3} = 1 - \dfrac{27J^2}{I^3}$.

Suppose, for shortness,

$$M = \frac{27}{4} \cdot \frac{1}{\left(1 - \frac{27J^2}{I^3}\right)}, \quad \text{or } \frac{1}{108}\left(1 - \frac{27J^2}{I^3}\right) = \frac{1}{16M},$$

$$(q^2 + 14q + 1)^3 - 16Mq(q-1)^4 = 0, \quad \textit{i.e.}$$

$$\left(q + \frac{1}{q} + 14\right)^3 - 16M\left(q^{\frac{1}{2}} - \frac{1}{q^{\frac{1}{2}}}\right)^4 = 0.$$

Let $\qquad q^{\frac{1}{2}} - q^{-\frac{1}{2}} = \dfrac{4}{(\theta - 1)^{\frac{1}{2}}}$,

then $\qquad \theta^3 - M(\theta - 1) = 0$,

which determines θ. And then

$$q = \frac{7 + \theta + 4\,(3+\theta)^{\frac{1}{2}}}{\theta - 1}.$$

Suppose $q = a$ is one of the values of q; the equation becomes

$$\frac{(q^2 + 14q + 1)^3}{q \cdot (q-1)^4} = \frac{(a^2 + 14a + 1)^3}{a\,(a-1)^4}$$

$$= \frac{(\beta^8 + 14\beta^4 + 1)^3}{\beta^4 \cdot (\beta^4 - 1)^4}, \quad \text{if } a = \beta^4.$$

Now if $q = \left(\frac{1-\beta}{1+\beta}\right)^4$,

$$(q^2 + 14q + 1) = \frac{16(\beta^8 + 14\beta^4 + 1)}{(1+\beta)^8}, \quad q - 1 = -\frac{8\beta(1+\beta^2)}{(1+\beta)^4},$$

which satisfy the above equation: hence also, identically,

$$(q^2 + 14q + 1)^3 - q(q-1)^4 . \frac{(\beta^8 + 14\beta^4 + 1)^3}{\beta^4 . (\beta^4 - 1)^4}$$

$$= (q - \beta^4)\left(q - \frac{1}{\beta^4}\right)\left\{q - \left(\frac{1-\beta}{1+\beta}\right)^4\right\}\left\{q - \left(\frac{1+\beta}{1-\beta}\right)^4\right\}$$

$$\left\{q - \left(\frac{1-\beta i}{1+\beta i}\right)^4\right\}\left\{q - \left(\frac{1+\beta i}{1-\beta i}\right)^4\right\};$$

or the values of q take the form

$$\beta^4, \frac{1}{\beta^4}, \left(\frac{1-\beta}{1+\beta}\right)^4, \left(\frac{1+\beta}{1-\beta}\right)^4, \left(\frac{1-\beta i}{1+\beta i}\right)^4, \left(\frac{1+\beta i}{1-\beta i}\right)^4.$$

(Comp. *Abel. Œuv.* tom. I. p. 310).

The equation $\theta^3 - M\theta + M = 0$

has its three roots real if $27 - 4M$ is negative, and only a single real root if $27 - 4M$ is positive. Writing the equation under the form

$$(\theta + 3)^3 - 9(\theta + 3)^2 + (27 - M)(\theta + 3) - (27 - 4M) = 0,$$

we see that in the former case θ has two values greater than -3, and a single value less than -3. Writing the equation under the form

$$(\theta - 1)^3 + 3(\theta - 1)^2 + (3 - M)(\theta - 1) + 1 = 0, \quad (3 - M \text{ is negative})$$

the positive roots are both greater than 1. Hence, in this case, q has four positive values and two imaginary ones. In the second case θ has a single real value, which is greater than -3 and less than 1. Hence q has two negative values and four imaginary ones. In the former case, $I^3 - 27J^2$ is positive, and the function U has either four imaginary factors or four real ones. In the second case, $I^3 - 27J^2$ is negative, or the function U has two real and two imaginary factors.

NOTE ON THE MAXIMA AND MINIMA OF FUNCTIONS OF THREE VARIABLES.

By Arthur Cayley, M.A., Fellow of Trinity College.

If $A, B, C, F, G, H,$ be any real quantities, such that

$$BC + CA + AB - F^2 - G^2 - H^2,$$

and $(A + B + C)(ABC - AF^2 - BG^2 - CH^2 + 2FGH)$

are positive; the six quantities

$$BC - F^2,\ CA - G^2,\ AB - H^2,\ AK,\ BK,\ CK,$$

(where $K = ABC - AF^2 - BG^2 - CH^2 + 2FGH$)

are all of them positive. It is unnecessary to point out the connection of this property with the theory of maxima and minima.

To demonstrate this, writing as usual

$$\begin{array}{ll} BC - F^2 = A', & GH - AF = F', \\ CA - G^2 = B', & HF - BG = G', \\ AB - H^2 = C', & FG - CH = H', \end{array}$$

and K as above: then if $A'', B'', C'', F'', G'', H'', K'$ be formed from A', B', C', F', G', H', as these and K are from $A, B, C, F, G, H,$ we have the well known formulæ

$$\begin{array}{lll} A'' = KA, & F'' = KF, & K' = K^2. \\ B'' = KB, & G'' = KG, & \\ C'' = KC, & H'' = KH, & \end{array}$$

It is required to show that if $A' + B' + C'$ and $A'' + B'' + C''$ are positive, $A', B', C', A'', B'', C''$ are so likewise.

Consider the cubic equation

$$(A'-k)(B'-k)(C'-k) - (A'-k)F'^2 - (B'-k)G'^2 - (C'-k)H'^2 + 2F'G'H' = 0,$$

the roots of which are all real. By the formulæ just given this may be written

$$k^3 - k^2(A' + B' + C') + k\,.\,(A'' + B'' + C'') - K^2 = 0;$$

and the terms of this equation are alternately positive and negative; *i.e.* the roots are all positive. Hence the roots of the limiting equation

$$(B' - k)(C' - k) - F'^2 = 0$$

are positive, *i.e.* $B' + C'$ and $B'C'$ are positive: but from the second condition B', C' are of the same sign. Consequently of the same sign with $B' + C'$ or positive. Also $A'' = B'C' - F'^2$

is positive. Similarly, considering the other limiting equations, A', B', C', A'', B'', C'' are all of them positive.

In connection with the above I may notice the following theorem. The roots of the equation

$$(A - ka)(B - kb)(C - ck) - (A - ka)(F - kf)^2 \\ - (B - kb)(G - kg)^2 - (C - kc)(H - kh)^2 \\ + 2(F - kf)(G - kg)(H - kh) = 0,$$

are all of them real, if either of the functions

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy,$$
$$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy,$$

preserve constantly the same sign. The above form parts of a general system of properties of functions of the second order.

ON THE MATHEMATICAL THEORY OF ELECTRICITY IN EQUILIBRIUM.

By WILLIAM THOMSON, B.A., Fellow of St. Peter's College.

I. *On the Elementary Laws of Statical Electricity.**

1. The elementary laws which regulate the distribution of electricity on conducting bodies have been determined by means of direct experiments, by Coulomb, and in the form he has given them, which is independent of any hypothesis,† they have long been considered as rigorously established. The problem of the distribution of electricity in equilibrium on a conductor of any form was thus brought within the province of mathematical analysis; but the solution, even in the simplest cases, presented so much difficulty that Coulomb, after having investigated it experimentally for bodies of various forms, could only compare his measurements with the results of his theory by very rude processes of approximation. Without however giving rigorous solutions in particular cases, he examined the general problem with great care, and left nothing indefinite in the conditions to be satisfied, so that it was entirely by analytical difficulties that he was stopped As an example of the success of his theoretical investigations, we may refer to the well-known demonstration of the theorem (usually attributed to Laplace) relative to the

* This paper is a translation (with considerable additions) of one which appeared in Liouville's *Journal de Mathématiques*, vol. x. p. 209.

† See the first Note at the end of this paper.

repulsion exercised by a charged conductor on a point near its surface.*

The memoirs of Poisson, on the mathematical theory, contain the analytical determination of the distribution of electricity on two conducting spheres placed near one another, the solution being worked out in numbers in the case of two equal spheres in contact, which had been investigated experimentally by Coulomb (as well as in another case, not examined by Coulomb, which is given as a specimen of the numerical results that may be deduced from the formulæ). The calculated ratios of the intensities at different points of the surface he is therefore enabled to compare with Coulomb's measurements, and he finds an agreement which is quite as close as could be expected, when we consider the excessively difficult and precarious nature of quantitative experiments in electricity: but the most remarkable confirmation of the theory from these researches is the entire agreement of the principal features, even in some very singular phenomena, of the experimental results with the theoretical deductions. For a complete account of the experiments we must refer to Coulomb's fifth memoir (*Histoire de l' Académie*, 1787), and for the mathematical investigations to the first and second memoirs of Poisson (*Mémoires de l' Institut*, 1811), or to the treatise on Electricity in the *Encyclopædia Metropolitana*, where the substance of Poisson's first memoir is given.

The mathematical theory received by far the most complete development which it has hitherto obtained, in Green's *Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*,† in which a series of general theorems were demonstrated, and many interesting applications made to actual problems.‡

Of late years some distinguished experimentalists have begun to doubt the truth of the laws established by Coulomb, and have made extensive researches with a view to discover the laws of certain phenomena which they considered incompatible with his theory. The most remarkable works of this kind have been undertaken independently by Mr. Snow Harris and Mr. Faraday, and in their memoirs, published in the *Philosophical Transactions*, we find detailed accounts of their researches. All the experiments, however, which they have made, having direct reference to the distribution of electricity in equilibrium, are, I think, in full accordance with the laws of Coulomb, and must therefore, instead of objections to his theory, be considered as confirming it. As however many

* See Note II. † Nottingham, 1828. ‡ See Note III.

have believed Coulomb's theory to be overturned by these investigations, and as others have at least been led to entertain doubts as to its certainty or accuracy, the following attempt to explain the apparent difficulties is made the subject of the first of a series of papers in which various parts of the mathematical theory of electricity, and corresponding problems in the theories of magnetism and heat, will be considered.

2. We may commence by examining some experimental results published in Mr. Harris's first memoir *On the Elementary Laws of Electricity*.* After describing the instruments employed in his researches, Mr. Harris gives the details of some experiments with reference to the attraction exercised by an insulated electrified body on an uninsulated conductor placed in its neighbourhood. The first result which he announces is that, when other circumstances remain the same, the attraction varies as the square of the quantity of electricity with which the insulated body is charged. It is readily seen, as was first remarked by Dr. Whewell in his *Report on the Theories of Electricity, &c.*,† that this is a rigorous deduction from the mathematical theory, following from the fact that the quantity of electricity induced upon the uninsulated body is proportional to the charge on the electrified body by which it is attracted.

The remaining results have reference to the force of attraction at different distances, and with bodies of different forms opposed. As these are generally very irregular (such as "plane circular areas backed by small cones"), we should not, according to Coulomb's theory, expect any very simple laws, such as Mr. Harris discovers, to be rigorously true. Accordingly, though they are announced by him without restriction, we must examine whether the experiments from which they have been deduced are of a sufficiently comprehensive character to lead to any general conclusions with respect to electrical action. Now in the first place, we find that in all of them the attraction is "independent of the form of the unopposed parts" of the bodies, which will be the case only when the intensity of the induced electricity on the unopposed parts of the uninsulated body is insensible. According to the mathematical theory, and according to Mr. Faraday's researches "on induction in curved lines," which will be referred to below, the intensity never absolutely vanishes at any point of the uninsulated body: but it is readily seen that in the case of

* Philosophical Transactions, 1834.
† British Association Report for 1837.

Mr. Harris's experiments, it will be so slight on the unopposed portions that it could not be perceived without experiments of a very refined nature, such as might be made by the proof plane of Coulomb, which is in fact, with a slight modification, the instrument employed by Mr. Faraday in the investigation. Now to the degree of approximation to which the intensity on the unopposed parts may be neglected, the laws observed by Mr. Harris when the opposed surfaces are plane may be readily deduced from the mathematical theory. Thus let v be the potential in the interior of the charged body, A, a quantity which will depend solely on the state of the interior coating of the battery with which in Mr. Harris's experiments A is connected, and will therefore be sensibly constant for different positions of A relative to the uninsulated opposed body, B. Let a be the distance between the plane opposed faces of A and B, and let S be the area of the opposed parts of these faces, which will in general be the area of the smaller, if they be unequal. When the distance a is so small that we may entirely neglect the intensity on all the unopposed parts of the bodies, it is readily shewn from the mathematical theory that (since the difference of the potentials at the surfaces of A and B is v) the intensity of the electricity produced by induction at any point of the portion of the surface of B which is opposed to A, is $\frac{v}{4\pi a}$, the intensity at any point which is not so situated being insensible. Hence the attraction on any small element ω, of the portion S of the surface of B, will be in a direction perpendicular to the plane and equal to $2\pi\left(\frac{v}{4\pi a}\right)^2$.* Hence the whole attraction on B is

$$\frac{v^2 S}{8\pi a^2}.$$

This formula expresses all the laws stated by Mr. Harris as results of his experiments in the case when the opposed surfaces are plane.

3. When the opposed surfaces are curved, for instance when A and B are equal spheres, we can make no approximation analogous to that which has led us to so simple an expression in the case of opposed planes; and we find accordingly that no such simple law for the attraction in this case has been announced by Mr. Harris. He has however found

* See *Mathematical Journal*, vol. III. p. 275.

that it is expressed with tolerable accuracy by the formula

$$F = \frac{k}{c\,(c - 2a)},$$

where c is the distance between the centres of the spheres, a the radius of each, k a constant, which will depend on a and on the charge of the battery with which A is in communication. Though however this formula may give results which do not differ very much from observation within a limited range of distances, it cannot, according to any theory, be considered as expressing the physical law of the phenomenon. For, according to it, when the balls are very distant, F ultimately varies as $\frac{1}{c^2}$. Now it is clear that the law of force must ultimately become the inverse cube of the distance, since the quantity of electricity induced upon B will be ultimately in the inverse ratio of the distance, and the attraction between the balls as the product of the quantities of electricity directly, and as the square of the distance inversely, and hence the formula given by Mr. Harris cannot express the law of force when the balls are very distant. In the experiments by which his formula is tested, the force of attraction is measured by means of an ordinary balance and weights: the only comparison of results which he publishes is transcribed in the following table.

Dist. of Centres.	Measured Force in Grains.	Values of $\frac{15c_1\,(c_1 - 2)}{c\,(c - 2)}$.
$c_1 = 2.3$	15	15
$c_2 = 2.5$	8.25 +	8.28
$c_3 = 2.8$	4.6 +	4.62
$c_4 = 3.0$	3.5 −	3.45

From this table we see that the formula is verified in three cases to the extent of accuracy of the experiments. Comparisons extended to a much wider range of distances would be required to establish it, and it would be necessary to take precautions to prevent the experimental results from being influenced by disturbing causes. In the experiments made by Mr. Harris we find that no precautions have been taken to avoid the disturbing influence of extraneous conductors, which, according to the descriptions and drawings he gives of his instruments, seem to exist very abundantly in the neighbourhood

of the bodies operated upon, being partly metal in connection with the insulated system with which the body A communicates, and partly uninsulated metal, in the fixed parts of the electrometer, and in the moveable parts by which B is supported. The general effect produced by the presence of such bodies in disturbing the observed law of force, must be to make it diminish less rapidly with the distance when A and B are separated by a considerable interval: and it is probably owing, at least in part, to such disturbing causes that Mr. Harris's results nearly agree, as far as they go, with a formula which would ultimately give for the law of force the inverse square of the distance between A and B, instead of the inverse cube.

4. The determination by the mathematical theory of the attraction or repulsion between two electrified conducting spheres has not hitherto, so far as I am aware, been attempted, and would present considerable difficulty by means of the formulæ ordinarily given for such problems. It may, however, very readily be effected by means of a general theorem on the attraction between electrified conductors, which will be given in a subsequent paper. Thus, if $F(c)$ be the force of attraction, corresponding to the distance c between the centres, in the particular case when the two spheres are equal (the radius of each being unity), and the potential in the interior of one of them is nothing (as will be the case when the body is uninsulated), the potential in the interior of the other being v, I have found the following formulæ which express $F(c)$ by a converging series.

$$\text{(A)} \qquad F(c) = v^2 c\left(\frac{P_1}{Q_1^2} + \frac{P_2}{Q_2^2} + \frac{P_3}{Q_3^2} + \&\text{c.}\right), \text{ where}$$

$$\text{(B)}, \qquad \begin{cases} Q_1 = c^2 - 1, \\ Q_2 = (c^2 - 2)\, Q_1 - 1, \\ Q_{n+2} = (c^2 - 2)\, Q_{n+1} - Q_n. \end{cases}$$

$$\text{(C)}, \qquad \begin{cases} P_1 = 1, \\ P_2 = 2c^2 - 3. \\ P_{n+2} = (c^2 - 2)\, P_{n+1} + (Q_{n+1} - P_n). \end{cases}$$

These formulæ enable us to calculate Q_1, Q_2, Q_3, Q_4, &c., and then P_1, P_2, P_3, P_4, &c., successively, by a simple and uniform arithmetical process, for any particular value of c. I have thus calculated the values of $\frac{F(c)}{v^2}$ in five cases, the

first four of which are those examined by Mr. Harris, and have obtained the following results, each of which is true to five places of decimals.

c	$v^{-2}F(c)$.
2·3	0·32926
2·5	0·17423
2·8	0·09168
3·0	0·06592
4·0	0·02075

To compare these with Mr. Harris's measurements we may calculate the value of the potential in his battery, during the observations, by means of his first result, and thence find the attraction for the other three cases by means of the calculated values of $v^{-2}F(c)$. Thus we have $v^{-2} \times 15 = 3293$, which gives

$$v^2 = 45{\cdot}56,$$

and hence

$$F(2{\cdot}5) = 7{\cdot}94,$$
$$F(2{\cdot}8) = 4{\cdot}18,$$
$$F(3) = 3{\cdot}00.$$

These numbers differ considerably from Mr. Harris's results, but in the direction indicated by the considerations mentioned above.

5. The most important part of the researches of Mr. Harris is that in which he investigates the insulating power of air of different densities. The result at which he arrives is, that the intensity necessary to produce a spark depends solely on the density of the air, and not otherwise on the pressure or temperature. He thus shews that the conducting power of flame, of heated bodies, and of a vacuum, are due solely to the rarefaction of the air in each case. He also shews that the intensities necessary to produce a spark, are in the simple ratios of the densities of the air.

6. In a subsequent memoir, by the same author,* we find additional experiments on the elementary principles of the theory of electricity. The first series which is described, was made for the purpose of testing the truth of Coulomb's law, that the repulsion of two similarly charged points is inversely as the square of the distance, and directly as the product of the masses. In experiments of this kind in which accurate quantitative results are aimed at, many precautions are ne-

* *Philosophical Transactions*, 1836.

cessary. Thus all conducting bodies except those operated upon, must be placed beyond the reach of influence, and the distance between the repelling bodies must be considerable with reference to their linear dimensions, so that the distribution of electricity on each may be uninfluenced by the presence of the other. Also the bodies should be spheres, so that the attraction may be the same as if the whole electricity of each were collected at its centre; and the distance to be measured will then be the distance between the centres. These conditions have been expressly mentioned by Coulomb, and they have been fulfilled, as far as possible, in his researches, as we see by the descriptions of the experiments made, which we find in his memoirs. He has thus arrived by direct measurement at the law, which we know by a mathematical demonstration,* founded upon independent experiments, to be the rigorous law of nature, for electrical action. None of these precautions however have been taken in the experiments described in Mr. Harris's memoir, and the results are accordingly unavailable for the accurate *quantitative* verification of any law, on account of the numerous unknown disturbing circumstances by which they are affected. The phenomena which he observes, however, afford *qualitative* illustrations of the mathematical theory of a very interesting nature, as may be seen from the following examples of his results.

(*a*) When the distance between the bodies is great with reference to their linear dimensions, the repulsion is inversely as the square of the distance, and directly as the product of the masses.

(*b*) When the distance is small, the action becomes apparently irregular. Thus if the quantities of electricity on the two bodies be equal, the force, which is always of repulsion, does not increase so rapidly when the bodies approach, as if it followed the law of the inverse square of the distance.

(*c*) If the charges be unequal, the repulsion ceases at a certain distance, and at all smaller distances there is attraction between the bodies.

These results are, with all their peculiarities, in full accordance with the theory of Coulomb, which indicates that, if the quantities of electricity be equal, and the bodies equal and similar, there will be repulsion in every position: but if there be any difference, however small, between the charges, the repulsion will necessarily cease, and attraction commence, before contact takes place, when one body is made to approach the other. Unless, however, the difference of the charges

* See Murphy's *Electricity*, p. 41, or Pratt's *Mechanics*, Art. 154.

be sufficiently considerable, a spark may pass between the bodies, and render the charges equal, before attraction commences. In Mr. Harris's experiments, in which the bodies seem to have been nearly oblate spheroids, the attraction is generally sensible before the distance is small enough to allow a spark to pass, if the charge on one be double of that on the other.

Mr. Harris next proceeds to investigate the theory of the proof plane, and to examine whether it can be considered as indicating with certainty the intensity of electricity at any part of a charged body, and, principally from an experiment made on a charged non-conductor (a hollow sphere of glass), comes to a negative conclusion. It should be remembered, however, that, the proof plane having never been applied to determine the intensity at points of the surface of a charged non-conductor, such conclusions in no way interfere with adopted ideas. Since there can be no manner of doubt as to the theory of this valuable instrument, as we find it explained by M. Pouillet,* nor as to the experimental use of it made by Coulomb, it is unnecessary to enter more at length on the subject here.

7. Mr. Faraday's researches on electrostatical induction, which are published in a memoir forming the eleventh series of his Experimental Researches in Electricity, were undertaken with a view to test an idea which he had long possessed, that the forces of attraction and repulsion exercised by free electricity, are not the resultant of actions exercised at a distance, but are propagated by means of molecular action among the contiguous particles of the insulating medium surrounding the electrified bodies, which he therefore calls the *dielectric*. By this idea he has been led to some very remarkable views upon induction, or in fact upon electrical action in general. As it is impossible that the phenomena observed by Faraday can be incompatible with the results of experiment which constitute Coulomb's theory, it is to be expected that the difference of his ideas from those of Coulomb must arise solely from a different method of stating, and interpreting physically, the same laws: and farther, it may I think be shewn that either method of viewing the subject, when carried sufficiently far, may be made the foundation of a mathematical theory which would lead to the elementary principles of the other as consequences. This theory would accordingly be the expression of the ultimate law of the phenomena, inde-

* See Note IV.

pendently of any physical hypothesis we might, from other circumstances, be led to adopt. That there are necessarily two distinct elementary ways of viewing the theory of electricity, may be seen from the following considerations, founded on the principles developed in a previous paper in this Journal.*

Corresponding to every problem relative to the distribution of electricity on conductors, or to forces of attraction and repulsion exercised by electrified bodies, there is a problem in the uniform motion of heat which presents the same analytical conditions, and which therefore, considered mathematically, is the same problem. Thus, let a conductor A, charged with a given quantity of electricity, be insulated in a hollow conducting shell, B, which we may suppose to be uninsulated. According to the mathematical theory, an equal quantity of electricity of the contrary kind will be attracted to the interior surface of B, (or the surface of B, as we may call it to avoid circumlocution), and the distribution of this charge, and of the charge on A, will take place so that the resultant attraction at any point of each surface may be in the direction of the normal. This condition being satisfied, it will follow that there is no attraction on any point within A, or without the surface of B, that is, on any point within either of the conducting bodies. The most convenient mathematical expression for the condition of equilibrium, is that the potential at any point P† must have a constant value when P is on the surface of A, and the value nothing when P is on the surface of B; and it will follow from this that the potential will have the same constant value for any point within A, and will be equal to nothing for any point without the surface of B.

If A be subject to the influence of any uninsulated conductors, we must consider such bodies as belonging to the shell in which A is contained, and their surfaces as forming part of the surface of B: in such cases this surface will generally be the interior surface of the walls of the room in which A is contained, and of all uninsulated conductors in the room. If however we have to consider the case in which A is subject to no external influence, we must suppose every part of the surface of B to be very far from A. The most general problem we can contemplate in electricity

* On the Uniform Motion of Heat, and its Connection with the Mathematical Theory of Electricity, Vol. III. p. 73.

† The term used by Green for the sum of the quotients obtained by dividing the product of each element of the surfaces of A and B, and its electrical intensity, by its distance from P.

(exclusively of the case in which the insulating medium is heterogeneous, and exercises a special action, which will be alluded to below), is to determine the potential at any point when A, instead of being a single conductor, is a group of separate insulated conductors charged to different degrees, and when there are non-conductors electrified in a given manner, placed in the insulating medium, in the neighbourhood. The conditions of equilibrium will still be that the potential at each surface due to all the free electricity must be constant, and the theorems stated above will still be true: thus the attraction will be nothing in the interior of each portion of A, and without the surface of B; and the whole quantity of induced electricity on the latter surface will be the algebraic sum of the charges of all the interior bodies with its sign changed. When the potential due to such a system is determined for every point, the component of the resultant force at any point P, in any direction PL, may be found by differentiation, being the limit of the difference between the values of the potential at P, and at a point Q, in PL, divided by PQ, when P moves up towards and ultimately coincides with P, and the direction of the force, on a *negative* particle, being that in which the potential increases. By Coulomb's theorem, the intensity at any point in one of the conducting surfaces is equal to the attraction (on a negative unit) at that point, divided by 4π.

Now if we wish to consider the corresponding problem in the theory of heat, we must suppose the space between A and B, instead of being filled with a dielectric medium (that is a non-conductor for electricity), to be occupied by any homogeneous solid body, and sources of heat or cold to be so distributed over the terminating surfaces, or the interior surface of B and the surface of A, that the permanent temperature at the first surface may be zero, and at the second shall have a certain constant value, the same as that of the *potential* in the case of electricity. If A consist of different isolated portions, the temperature at the surface of each will have a constant value, which is not necessarily the same for the different portions. The problem of *distributing sources of heat, according to these conditions,* is mathematically identical with the problem of distributing *electricity in equilibrium* on the surfaces of A and B. In the case of heat, the *permanent temperature* at any point replaces the *potential* at the corresponding point in the electrical system, and consequently the *resultant flux of heat* replaces the *resultant attraction* of the electrified bodies, in direction and magnitude. The

problem in each case is determinate, and we may therefore employ the elementary principles of one theory, as theorems, relative to the other. Thus, in the paper in which these considerations are developed, Coulomb's fundamental theorem relative to electricity is applied to the theory of heat; and self-evident propositions in the latter theory are made the foundation of Green's theorems in electricity.* Now the laws of motion for heat which Fourier lays down in his *Théorie Analytique de la Chaleur*, are of that simple elementary kind which constitute a mathematical theory properly so called; and therefore, when we find corresponding laws to be true for the phenomena presented by electrified bodies, we may make them the foundation of the mathematical theory of electricity: and this may be done if we consider them merely as actual truths, without adopting any physical hypothesis, although the idea they naturally suggest is that of the propagation of some effect by means of the mutual action of contiguous particles; just as Coulomb, although his laws naturally suggest the idea of material particles attracting or repelling one another at a distance, most carefully avoids making this a *physical hypothesis*, and confines himself to the consideration of the mechanical effects which he observes and their necessary consequences.†

All the views which Faraday has brought forward, and illustrated or demonstrated by experiment, lead to this method of establishing the mathematical theory, and, as far as the analysis is concerned, it would, in most *general* propositions, be even more simple, if possible, than that of Coulomb. (Of course the analysis of *particular* problems would be identical in the two methods). It is thus that Faraday arrives at a knowledge of some of the most important of the general theorems, which, from their nature, seemed destined never to be perceived except as mathematical truths. Thus, in his theory, the following proposition is an elementary principle. Let any portion α of the surface of A be projected on B, by means of lines (which will be in general curved) possessing the property that the resultant electrical force at any point of each of them is in the direction of the tangent: the quantity of electricity produced by induction on this projection is equal to the quantity of the opposite kind of elec-

* It was not until some time after that paper was published, that I was able to add the direct analytical demonstrations of the theorems, which are given in the papers on "General Propositions in the Theory of Attraction," vol. III. pp. 189, 201, and which I have since found are the same as those originally given by Green.

† See Note I.

tricity on a.* The lines thus defined are what Faraday calls the "curved lines of inductive action." For a detailed account of the experiments by which these phenomena are investigated, reference must be made to Mr. Faraday's own memoirs, published in the Philosophical Transactions, and in a separate form in his Experimental Researches.

8. The hypothesis adopted by Faraday, of the *propagation* of inductive action, naturally led him to the idea that its effects may be in some degree dependent upon the nature of the insulating medium or dielectric, by which, according to this view, it is transmitted. In the second part of his memoir he describes a series of researches instituted to put this to the test of experiment, and arrives at the following conclusions.

If the dielectric be air, the inductive action is quite independent of its density or temperature (which, as Mr. Faraday remarks, agrees perfectly with previous results obtained by Mr. Harris); and in general, if the dielectric be any gas or vapour capable of insulating a charge, the inductive action is invariable. Hence he concludes that "*all gases have the same power of*, or *capacity for*, sustaining induction through them, (which might have been expected when it was found that no variation of density or pressure produced any effect.)"

When the dielectric is solid, the induction is greater than through air, and varies according to the nature of the substance. Numbers which measure the "specific inductive capacities" of the dielectrics employed (sulphur, shell lac, glass, &c.), are deduced from the experiments.

To express these results in the language of the mathematical theory, let us recur to the supposition of a body, A, charged with a given quantity of electricity, and insulated in the interior of a closed conducting shell, B. The potential of the system at the interior surface of B, and at every point without this surface, will be nothing; at the surface and in the interior of A it will have a constant value, which will depend on the form, magnitude, and relative position of the surfaces A and B, on the quantity of electricity on A, and, according to Faraday's discovery, on the *dielectric power* of the insulating medium which fills the space between A and B. If this be gaseous, neither its nature, nor its state as to temperature, pressure, or density, will affect the value of the potential in A; but if it be a solid substance, such as sulphur or shell lac, the value of the potential will be less than when the space is occupied by air, and will vary with the nature of the insulating solid.

* See Note IV.

The result in the case of a gaseous dielectric is what would follow from Coulomb's theory, if we consider gases to be quite impermeable to electricity, and to be entirely unaffected by electrical influence. The phenomena observed with solid dielectrics, which agree with the circumstance observed by Nicholson, that the *dissimulating power* of a Leyden phial depends on the nature of the glass of which it is made, as well as on its thickness, have been by some attributed to a slight degree of conducting power, or of penetrability, possessed by solid insulators. This explanation, however, seems to be very insufficient; and besides, Faraday has estimated the nature of the effects of imperfect insulation, by independent experiments, and has established, in what seems to be a very satisfactory manner, the existence of a peculiar action in the interior of solid insulators when subjected to electrical influence. As far as can be gathered from the experiments which have yet been made, it seems probable that a dielectric, subjected to electrical influence, becomes excited in such a manner that every portion of it, however small, possesses *polarity* exactly analogous to the magnetic polarity induced in the substance of a piece of soft iron under the influence of a magnet. By means of a certain hypothesis regarding the nature of magnetic action,* Poisson has investigated the mathematical laws of the distribution of magnetism and of magnetic attractions and repulsions. These laws seem to represent in the most general manner the state of a body polarized by influence, and therefore, without adopting any particular mechanical hypothesis, we may make use of them to form a mathematical theory of electrical influence in dielectrics, the truth of which can only be established by a rigorous comparison of its results with experiment.

Let us therefore consider what would be the effect, according to this theory, which would be produced by the presence of a solid dielectric, C, placed in the space between A and B, the rest of which is occupied by air. The action of C, when

* Faraday adopts the corresponding hypothesis to explain the action of a solid dielectric, which he states thus:—"If the space round a charged globe were filled with a mixture of an insulating dielectric, as oil of turpentine or air, and small globular conductors, as shot, the latter being at a little distance from each other, so as to be insulated, then these in their condition and action exactly resemble what I consider to be the condition and action of the particles of the insulating dielectric itself. If the globe were charged, these little conductors would all be polar; if the globe were discharged, they would all return to their normal state, to be polarized again upon the recharging of the globe." (*Experimental Researches*, §. 1679.) The results of the mathematical analysis of such an action are given in the text. It may be added that the value of the coefficient k will differ sensibly from unity if the volume occupied by the small conducting balls bear a finite ratio to that occupied by the insulating medium.

excited by the influence of the electricities on A and B, may (as Poisson has shewn for magnetism) be represented, whether on points within or without C, by a certain distribution or positive electricity on one portion of the surface of C, and of an equal quantity of negative electricity on the remainder. The condition necessary and sufficient for determining this distribution may (as can be shewn from Poisson's analysis) be expressed as follows. Let R be the resultant force on a point P without C, and R' on a point P' within C, due to the electrified surfaces A and B, and to the imagined distribution on C. If P and P' be taken infinitely near one another, and consequently each infinitely near the surface of C, the component of R' in the direction of the normal must bear to the component of R in the same direction a constant ratio $\left(\frac{1}{k}\right)$ depending on the capacity for dielectric induction of the matter of C.* The components of R and R' in the tangent plane will of course be equal and in the same direction, and, if ρ be the intensity of the imagined distribution on the surface of C, in the neighbourhood of P and P', the difference of the normal components will be $4\pi\rho$, as is evident from Coulomb's theorem, referred to above.

Let us now suppose C to be a shell surrounding A, and let S and S', its interior and exterior surfaces, be *surfaces of equilibrium* in the system of forces due to the action of A and B, and of the polarity of C. It may be shewn that the same surfaces S, S', would necessarily be surfaces of equilibrium, if C were removed and the whole space were filled with air; and consequently, that the whole series of surfaces of equilibrium, commencing with A and ending with B, will be the same in the two cases. Hence the resultant force due to the excitation of the dielectric C, or to the imagined distributions of electricity on S and S' which produce it, on points within S or without S', must be such as not to alter the distributions on A and B when the quantity on A is given, and is therefore nothing. Accordingly, let Q be the total force on a point indefinitely near S, and within it; Q' the total force on a point without S', but indefinitely near it. Since the forces on points without S and within S' indefinitely near the former points are, according to the law stated above, $\frac{Q}{k}$ and $\frac{Q'}{k}$, it follows that the intensities of the imagined distributions on

* From this it follows that, in the case of heat, C must be replaced by a body whose conducting power is k times as great as that of the matter occupying the remainder of the space between A and B.

S and S', in the neighbourhood of the points considered, are

$$-\frac{1}{4\pi}\left(Q - \frac{Q}{k}\right) \text{ and } \frac{1}{4\pi}\left(Q' - \frac{Q'}{k}\right).$$

Hence, if U, U' be the potentials at S, S', due to A and B alone, and v the potential at any point P, it follows* that the potential at P, due to the polarity of the dielectric, is

$$-\left(1 - \frac{1}{k}\right) U + \left(1 - \frac{1}{k}\right) U',$$

or
$$-\left(1 - \frac{1}{k}\right) v + \left(1 - \frac{1}{k}\right) U',$$

or
$$-\left(1 - \frac{1}{k}\right) v + \left(1 - \frac{1}{k}\right) v, \text{ that is, } 0,$$

according as P is within S, within S' and without S, or without S'. Hence the total potential will be, according to the position of P,

$$v - \left(1 - \frac{1}{k}\right)(U - U'),$$

or
$$\frac{v}{k} + \left(1 - \frac{1}{k}\right) U',$$

or
$$v.$$

Hence the sole effect of the dielectric C, on the state of A and B, is to diminish the potential in the interior of the former by the quantity

$$\left(1 - \frac{1}{k}\right)(U - U').$$

If the whole space between A and B be occupied by the solid dielectric, the surfaces S and A will coincide, as also, S' and B, and therefore $U = V$, $U' = 0$. Hence the potential in the interior of A will be

$$\frac{V}{k},$$

or the fraction $\frac{1}{k}$ of the potential, with the same charge on A, and with a gaseous dielectric. From this it follows that, when the dielectric is solid, it would require, to produce a given potential in the interior of A, k times the charge which would be necessary to produce the same potential when the dielectric is gaseous, and therefore the body A in a given state, defined by the potential in its interior, produces on the interior surface of B, by induction, through the solid dielectric, a quantity of electricity k times as great as through a gaseous dielectric. On this account Faraday calls the property of a dielectric measured by k, its "specific inductive capacity."

* See Green's *Essay*, Art. 12; or, *Math. Journal*, vol. III. p. 75.

In Faraday's experiments an apparatus (which is in fact a Leyden phial, in which any solid or fluid may be substituted for the glass dielectric of an ordinary Leyden phial) is used, corresponding to the case we have been considering, in which A is a conducting sphere (2.33 inches in diameter), and B a concentric spherical shell surrounding it (the distance between the surfaces of A and B being ·62 of an inch). In the shell B there is an aperture into which a shell lac stem is fixed; a wire, attached to A, passes through the centre of this stem to the outside of the shell, and supports a ball of metal, M, which is thus insulated and connected with A. It may be shewn that in such an apparatus the state of the ball A and of the shell B will approximately be not affected by the aperture in the latter, or by the wire supporting M, and that the distribution of electricity on M will be approximately the same as if the wire supporting it and the conductors A and B were removed. Hence the sole relation between A and M will be that the *potentials* in their interiors are the same; and therefore the latter, which is accessible, may be taken as an index of the state of the former.

To determine the specific inductive capacity of any dielectric, Faraday uses two apparatus of the kind just described, precisely equal and similar, in one of which the space between A and B is filled with air, and in the other with the dielectric to be examined. One of these apparatus is charged, and the intensity measured: the balls M, M' in the two are then made to touch and separated again, and the remaining intensity on the first (which is equal to the intensity imparted to the second) is measured. If this be found to differ from half the original intensity, it will follow that the specific inductive capacity of the substance examined differs from that of air, which is unity, and its value may be determined by means of a simple expression from the experimental data. To investigate this, let us first suppose each apparatus to be charged, and let it be required to find the intensity on the balls after they are made to touch, and then removed from mutual influence; and let the dielectrics be any two substances, whose inductive capacities are k, k'. Let ρ, ρ' be the intensities before, and σ the common intensity after contact. Then, denoting by Q, Q' the quantities of electricity constituting the charges before, and q, q' after contact, we shall have, by the principles already developed,

$$\frac{Q}{Q'} = \frac{k\rho}{k'\rho'}, \quad \frac{\sigma}{\rho} = \frac{q}{Q}, \quad \frac{\sigma}{\rho'} = \frac{q'}{Q'}.$$

Also $$Q + Q' = q + q'.$$

Hence we deduce $$\sigma = \frac{k\rho + k'\rho'}{k + k'}.$$

In the experiment described, one of the dielectrics is air. Hence, to obtain the required formula, we may put $k' = 1$, in this equation, and then resolve for k.

Thus we find $$k = \frac{\sigma - \rho'}{\rho - \sigma}.$$

If only one of the apparatus be originally charged, according as it is the first or the second, we shall have

$$k = \frac{\sigma}{\rho - \sigma},$$

or $$k = \frac{\rho' - \sigma}{\sigma}.$$

If the substance examined (the dielectric of the first apparatus) be any gas, or air in a different state as to pressure or temperature from the air of the second apparatus, Faraday always finds the intensity after contact to be half the original intensity, and hence for every gaseous body $k = 1$.

If the dielectric of the first apparatus be solid, the intensity after contact is found to be greater than half the original intensity when the first, and less than half when the second is the apparatus originally charged. Hence for a solid dielectric, $k > 1$. For sulphur Faraday finds the value to be rather more than 2·2; for shell-lac, about 2; and for flint-glass, greater than 1·76.

The commonly received ideas of attraction and repulsion exercised at a distance, independently of any intervening medium, are quite consistent with all the phenomena of electrical action which have been here adduced. Thus we may consider the particles of air in the neighbourhood of electrified bodies to be entirely uninfluenced, and therefore to produce no effect in the resultant action on any point: but the particles of a solid non-conductor must be considered as assuming a polarized state when under the influence of free electricity, so as to exercise attractions or repulsions on points at a distance, which, with the action due to the charged surfaces, produce the resultant force at any point. It is, no doubt, possible that such forces at a distance may be discovered to be produced entirely by the action of contiguous particles of some intervening medium, and we have an analogy for this in the case of heat, where certain effects which follow the same laws are undoubtedly propagated

from particle to particle. It might also be found that magnetic forces are propagated by means of a second medium, and the force of gravitation by means of a third. We know nothing however of the molecular action by which such effects could be produced, and in the present state of physical science it is necessary to admit the known facts in each theory as the foundation of the ultimate laws of action at a distance.

St. Peter's College, Nov. 22, 1845.

NOTES.

Note I.

Coulomb has expressed his theory in such a manner that it can only be attacked in the way of proving his experimental results to be inaccurate. This is shewn in the following remarkable passage in his sixth memoir, which follows a short discussion of some of the physical ideas then commonly held with reference to electricity. "*Je préviens pour mettre la théorie qui va suivre à l'abri de toute dispute systématique, que dans la supposition des deux fluides électriques, je n'ai d'autre intention que de présenter avec le moins d'élémens possible, les resultats du calcul et de l'expérience, et non d'indiquer les véritables causes de l'électricité. Je renverrai, à la fin de mon travail sur l'électricité, l'examen des principaux systèmes auxquels les phénomènes électriques ont donné naissance.*"—Histoire de l'Académie, 1788, p. 673.

Note II.

This theorem may be stated as follows. Let A be a closed surface of any form, and let matter, attracting inversely as the square of the distance, be so distributed over it that the resultant attraction on an interior point is nothing: the resultant attraction on an exterior point, indefinitely near any part of the surface, will be perpendicular to the surface and equal to $4\pi\rho$, if $\rho\omega$ be the quantity of matter on an element ω of the surface in the neighbourhood of the point. Coulomb's demonstration of this theorem may be found in a preceding paper in the *Mathematical Journal*, vol. III. p. 74. He gives it himself, in his sixth memoir on Electricity (*Histoire de l'Académie*, 1788, p. 677), in connection with an investigation of the theory of the proof plane in which, by an error that is readily rectified, he arrives at the result that a small insulated conducting disc, put in contact with an electrified conductor at any point, and then removed, carries with it as much electricity as lies on an element of the conductor at that point equal in area to the two faces of the disc; the quantity actually removed being only half of this. This result, however, does not at all affect the experimental use which he makes of the proof plane, which is merely to find the ratios of the intensities at different points of a charged conductor. As the complete theory of this valuable instrument has not, so far as I am aware, been given in any English work, I annex the following remarkably clear account of it, which is extracted from Pouillet's *Traité de Physique*:—"Quand le plan d'épreuve est tangent à une surface, il se confond avec l'élément qu'il touche, il prend en quelque sorte sa place relativement à l'électricité, ou plutôt il devient lui-même l'élément sur lequel la fluide se répand; ainsi, quand on retire

ce plan, on fait la même chose que si l'on avait découpé sur la surface un élément de même épaisseur et de même étendue que lui, et qu'on l'eût enlevé pour le porter dans la balance sans qu'il perdît rien de l'électricité qui le couvre; une fois séparé de la surface, cet élément n'aurait plus dans ses différents points qu'une épaisseur électrique moitié moindre, puisque la fluide devrait se répandre pour en couvrir les deux faces. Ce principe posé, l'expérience n'exige plus que de l'habitude et de la dextérité: après avoir touché un point de la surface avec le plan d'épreuve, on l'apporte dans la balance, où il partage son électricité avec le disque de l'aiguille qui lui est égale, et l'on observe la force de torsion à une distance connue. On répète la même experience en touchant un autre point, et le rapport des forces de torsion est le rapport des repulsions électriques; on en prend la racine carrée pour avoir le rapport des épaisseurs. Ainsi le génie de Coulomb a donné en même temps aux mathématiciens la loi fondamentale suivant laquelle la matière électrique s'attire et se repousse; et aux physiciens une balance nouvelle, et des principes d'expérience au moyen desquels ils peuvent en quelque sorte sonder l'épaisseur de l'électricité sur tous les corps, et determiner les pressions qu'elle exerce sur les obstacles qui l'arrêtent."

To this explanation it should be added that, when the proof plane is still very near the body to which it has been applied, the effect of mutual influence is such as to make the intensity be insensible at every point of the disc on the side next the conductor, and at each point of the conductor which is *under* the disc. It is only when the disc is removed to a considerable distance that the electricity spreads itself symmetrically on its two faces, and that the intensity at the point of the conductor to which it was applied, recovers its original value. It was the omission of this consideration that caused Coulomb to fall into the error alluded to above.

Note III.

This memoir of Green's has been unfortunately very little known, either in this country or on the continent. Some of the principal theorems in it have been re-discovered within the last few years, and published in the following works:—

Comptes Rendues for Feb. 11th, 1839, where part of the series of theorems is announced without demonstration, by Chasles.

Gauss's memoir on "General Theorems relating to Attractive and Repulsive Forces, varying inversely as the square of the distance," in the *Resultate aus den Beobachtungen des magnetischen Vereins im Jahre* 1839, *Leipsic*, 1840. (Translations of this paper have been published in *Taylor's Scientific Memoirs* for April 1842, and in the Numbers of *Liouville's Journal* for July and August 1842.)

Mathematical Journal, vol. III., Feb. 1842, in a paper "On the Uniform Motion of Heat, &c."

Additions to the Connaissance des Tems for 1845 (published June 1842), where Chasles supplies demonstrations of the theorems which he had previously announced.

I should add that it was not till the beginning of the present year (1845) that I succeeded in meeting with Green's Essay. The allusion made to his name with reference to the word "potential" (*Mathematical Journal*, vol. III. p. 190), was taken from a memoir of Murphy's, "On Definite Integrals with Physical Applications," in the *Cambridge Transactions*, where a mistaken definition of that term, as used by Green, is given.

NOTE IV.

This theorem may be proved as follows:—

Let S be any closed surface, containing no part of the electrified bodies within it, which we may conceive to be described between A and B; let P be the component in the direction of the normal, of the resultant force at any point of the surface S, and let ds be an element of the surface at the same point. Then it may be easily proved (see vol. III. p. 204), that

$$\iint Pds = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a),$$

the integrations being extended over the entire surface. Now let S be supposed to consist of three parts; the portion α, of the surface of A; its projection β, on the interior surface of B; and the surface generated by the curved lines of projection. The value of P at each point of the latter portion of S will be nothing, since the tangent at any point of a line of projection is the direction of the force. Hence, if $[\iint Pds]$, and $(\iint Pds)$ denote the values of $\iint Pds$, for the portions α and β of S, the equation (a) becomes

$$[\iint Pds] + (\iint Pds) = 0.$$

But if ρ be the intensity of the distribution on the surface A or B, at any point, we have, by Coulomb's theorem,

$$\rho = \frac{P}{4\pi}.$$

Hence $$[\iint \rho ds] + (\iint \rho ds) = 0,$$

which is the theorem quoted in the text.

MATHEMATICAL NOTES.

Solution of an Optical Problem proposed in the Senate-House Papers of 1844.

"IF a polished plane have an indefinite number of very fine concentric circular grooves turned on its surface, and light be incident on it from a luminous point, the appearance presented to the eye of an observer will be that of a bright curve; find its equation."

The solution depends very simply on the principle that, when a ray of light is reflected at any surface, the length of the course of the ray, reckoned from any point in the incident ray to any point in the reflected, is a minimum. For, in the above case, let the polished plane be the plane of xy, and the centre of the circular grooves the origin, $x_1 y_1 z_1$ the coordinates of the luminous point, $e f g$ those of the eye, and $x\, y\, 0$ those of the point of incidence on the plane corresponding to the groove whose radius is r; then the length of the path of the ray is

$$\sqrt{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}} + \sqrt{\{(e - x)^2 + (f - y)^2 + g^2\}},$$

which is to be a minimum subject to the condition

$$x^2 + y^2 = r^2,$$

or
$$\frac{dy}{dx} = -\frac{x}{y}.$$

Hence the condition of minimum is

$$\frac{(x_1 - x)\,y - (y_1 - y)\,x}{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}^{\frac{1}{2}}} + \frac{(e - x)\,y - (f - y)\,x}{(e - x)^2 + (f - y)^2 + g^2\}^{\frac{1}{2}}} = 0 \ldots (a),$$

or

$$\frac{x_1 y - y_1 x}{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}^{\frac{1}{2}}} + \frac{ey - fx}{\{(e - x)^2 + (f - y)^2 + g^2\}^{\frac{1}{2}}} = 0 \ldots (\text{A});$$

which, if x and y be taken as current coordinates, is the equation required. The radicals have been allowed to remain, because if they had been expelled by squaring, the result would have comprised the case in which the *difference* of the lengths of the incident and reflected ray is a minimum, and we should then have introduced a branch of the curve which is extraneous to the problem.

H. G.

[Equation (a) obviously expresses the condition, that straight lines drawn from any point P of the bright curve to the eye and to the luminous point, make equal angles with the tangent to the circle described from C as centre through P, and in the plane (x, y).

If light from a luminous point be incident upon a polished rod of any form, it would follow directly from the law of reflection, that a bright point will be seen on the rod in every position, such that lines drawn from it to the eye and to the luminous point, make equal angles with the tangent. From this we might immediately deduce the solution of the above problem as well as of the following.

A straight polished rod revolves rapidly in a plane about a fixed point; to find the bright curve which is seen by an eye in any position, when light is incident from a luminous point.

Taking, as in the preceding problem, $(x_1, y_1, z_1,)$ for the coordinates of the luminous point Q; $(e, f, g,)$ for those of the eye E, and $(x, y, 0,)$ for those of the image P of the luminous point seen in the rod at any instant, or, which is the same, of a point in the bright curve, we have, for the condition that QP and EP may be equally inclined to the rod OP,

$$\frac{(x_1 - x)\,x + (y_1 - y)\,y}{\{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2\}^{\frac{1}{2}}} + \frac{(e - x)\,x + (f - y)\,y}{\{(e - x)^2 + (f - y)^2 + g^2\}^{\frac{1}{2}}} = 0,$$

which is therefore the equation of the bright curve.]

ON HOMOGENEOUS FUNCTIONS OF THE THIRD ORDER WITH THREE VARIABLES.

By Arthur Cayley.

The following problem corresponds to the geometrical question of determining the polar reciprocal of a plane curve of the third order: the solution of it is also important, with reference to the linear transformations of homogeneous functions of three variables of the third order; reasons for which it has appeared to me worth while to obtain the completely developed result.

Let
$$3U = ax^3 + by^3 + cz^3 + 3iy^2z + 3jz^2x + 3kx^2y + 3i_1yz^2 + 3j_1zx^2 + 3k_1xy^2 + 6lxyz \;\ldots\ldots\; (1).$$

It is required to eliminate x, y, z, λ from the equations

$$U = 0 \;\ldots\ldots\ldots\ldots\ldots\; (2),$$

$$\left.\begin{aligned} \frac{dU}{dx} + \lambda\xi &= 0 \\ \frac{dU}{dy} + \lambda\eta &= 0 \\ \frac{dU}{dz} + \lambda\zeta &= 0 \end{aligned}\right\} \;\ldots\ldots\ldots\ldots\; (3).$$

From the equations (2), (3), we obtain immediately

$$\Theta = \xi x + \eta y + \zeta z = 0 \ldots\ldots\ldots\ldots (4);$$

and thence $\Theta x = 0, \quad \Theta y = 0, \quad \Theta z = 0 \;\ldots\ldots\; (5)$:

so that a single equation more, such as

$$\Phi = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where Φ is homogeneous and of the second order in x, y, z, would, in conjunction with the equations (3) and (5), enable us to eliminate linearly the seven quantities $x^2, y^2, z^2, yz, zx, xy, \lambda$. Such an equation may be thus obtained.

Let L, M, N, R, S, T, be the second differential coefficients of U, each of them divided by two. The equations (3) may be written

$$\begin{aligned} Lx + Ty + Sz + \lambda\xi &= 0, \;\ldots\ldots\ldots\; (7), \\ Tx + My + Rz + \lambda\eta &= 0, \\ Sx + Ry + Nz + \lambda\zeta &= 0. \end{aligned}$$

And joining to these the equation (4),

$$\xi x + \eta y + \zeta z = 0,$$

we have, by the elimination of x, y, z, in so far as they explicitly appear, and λ, an equation $\Phi = 0$ of the required form. Hence we may write

$$\Phi = -\begin{vmatrix} L, & T, & S, & \xi \\ T, & M, & R, & \eta \\ S, & R, & N, & \zeta \\ \xi, & \eta, & \zeta, & \end{vmatrix} \quad \ldots\ldots\ldots\ldots (8);$$

or substituting for L, M, N, R, S, T, and expanding,

$$\Phi = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy \ldots (9);$$

where

$$\left.\begin{aligned} A &= (k_1j - l^2)\,\xi^2 + (ja - j_1^2)\,\eta^2 + (ak_1 - k^2)\,\zeta^2 + 2\,(j_1k - al)\,\eta\zeta \\ &\qquad + 2\,(kl - k_1j_1)\,\zeta\xi + 2\,(lj_1 - jk)\,\eta\xi, \\ B &= (bi_1 - i^2)\,\xi^2 + (i_1k - l^2)\,\eta^2 + (bk - k_1^2)\,\zeta^2 + 2\,(lk_1 - ik)\,\eta\zeta \\ &\qquad + 2\,(k_1i - bl)\,\zeta\xi + 2\,(il - i_1k_1)\,\eta\xi, \\ C &= (ci - i_1^2)\,\xi^2 + (cj_1 - j^2)\,\eta^2 + (j_1i - l^2)\,\zeta^2 + 2\,(jl - j_1i_1)\,\eta\zeta \\ &\qquad + 2\,(li_1 - ij)\,\zeta\xi + 2\,(i_1j - cl)\,\eta\xi, \\ 2F &= (bc - ii_1)\,\xi^2 + (i_1j_1 + ck - 2lj)\,\eta^2 + (ki + bj_1 - 2lk_1)\,\zeta^2 \\ &\qquad + (l^2 + k_1j - ki_1 - j_1i)\,\eta\zeta + (k_1i_1 - bj)\,\zeta\xi + (ij - ck_1)\,\eta\xi, \\ 2G &= (ij + ck_1 - 2li_1)\,\xi^2 + (ca - jj_1)\,\eta^2 + (j_1k_1 + ai - 2lk)\,\zeta^2 \\ &\qquad + (jk - ai_1)\,\eta\zeta + (l^2 + i_1k - ij_1 - k_1j)\,\zeta\xi + (i_1j_1 - ck)\,\eta\xi, \\ 2H &= (k_1i_1 + bj - 2li)\,\xi^2 + (jk + ai_1 - 2lj_1)\,\eta^2 + (ab - kk_1)\,\zeta^2 \\ &\qquad + (j_1k_1 - ai)\,\eta\zeta + (ki - bj_1)\,\zeta\xi + (l^2 + j_1i - jk_1 - i_1k)\,\eta\xi, \end{aligned}\right\} \quad \ldots\ldots (10).$$

Performing the elimination indicated, the result may be represented by

$$\mathsf{F}U = \begin{vmatrix} a, & k_1, & j, & l, & j_1, & k, & \xi \\ k, & b, & i_1, & i, & l, & k_1, & \eta \\ j_1, & i, & c, & i_1, & j, & l, & \zeta \\ 2\xi, & . & . & . & \zeta, & \eta & . \\ . & 2\eta & . & \zeta & . & \xi & . \\ . & . & 2\zeta & \eta & \xi & . & . \\ A & B & C & F & G & H & . \end{vmatrix} = 0 \quad \ldots\ldots (11).$$

Partially expanding,

$$\mathsf{F}U = A\mathrm{a} + B\mathrm{b} + C\mathrm{c} + 2F\mathrm{f} + 2G\mathrm{g} + 2H\mathrm{h} \ldots (12).$$

The values of the coefficients a, b, c, f, g, h, may be useful on other occasions: they are as follows.

$$\left.\begin{aligned}
\mathrm{a} = {} & 0\xi^4 + 2(cj_1 - j^2)\eta^4 + 2(bk - k_1^2)\zeta^4 \\
& + 2(4jl - 3i_1j_1 - ck)\eta^3\zeta + 4(k_1i - lb)\zeta^3\xi + 0\xi^3\eta \\
& + 2(4k_1l - 3ik - j_1b)\eta\zeta^3 + 0\zeta\xi^3 + 4(i_1j - lc)\xi\eta^3 \\
& + 2(3i_1k + 3j_1i - 2jk_1 - 4l^2)\eta^2\zeta^2 + 2(bi_1 - i^2)\zeta^2\xi^2 \\
& \qquad + 2(ci - i_1^2)\xi^2\eta^2 \\
& + 2(ii_1 - bc)\xi^2\eta\zeta + 4(ck_1 + i_1l - 2ij)\xi\eta^2\zeta \\
& \qquad + 4(bj + il - 2i_1k_1)\xi\eta\zeta^2. \\
\mathrm{b} = {} & 2(ci - i_1^2)\xi^4 + 0\eta^4 + 2(ak_1 - k^2)\zeta^4 \\
& + 0\eta^3\zeta + 2(4kl - 3j_1k_1 - ai)\zeta^3\xi + 4(i_1j - lc)\xi^3\eta \\
& + 4(j_1k - al)\eta\zeta^3 + 2(4i_1l - 3ji - k_1c)\zeta\xi^3 + 0\xi\eta^3 \\
& + 2(aj - j_1^2)\eta^2\zeta^2 + 2(3j_1i + 3k_1j - 2ki_1 - 4l^2)\zeta^2\xi^2 \\
& \qquad + 2(cj_1 - j^2)\xi^2\eta^2 \\
& + 4(ck + jl - 2j_1i_1)\xi^2\eta\zeta + 2(jj_1 - ca)\xi\eta^2\zeta \\
& \qquad + 4(ai_1 + j_1l - 2jk)\xi\eta\zeta^2. \\
\mathrm{c} = {} & 2(bi_1 - i^2)\xi^4 + 2(aj - j_1^2)\eta^4 + 0\zeta^4 \\
& + 4(j_1k - al)\eta^3\zeta + 0\zeta^3\xi + 2(4il - 3k_1i_1 - bj)\xi^3\eta \\
& + 0\eta\zeta^3 + 4(k_1i - bl)\zeta\xi^3 + 2(4j_1l - 3kj - i_1a)\xi\eta^3 \\
& + 2(ak_1 - k^2)\eta^2\zeta^2 + 2(bk - k_1^2)\zeta^2\xi^2 \\
& \qquad + 2(3k_1j + 3i_1k - 2ij_1 - 4l^2)\xi^2\eta^2 \\
& + 4(bj_1 + k_1l - 2ki)\xi^2\eta\zeta + 4(ai + kl - 2k_1j_1)\xi\eta^2\zeta \\
& \qquad + 2(kk_1 - ab)\xi\eta\zeta^2. \\
\mathrm{f} = {} & (ii_1 - bc)\xi^4 + 0\eta^4 + 0\zeta^4 \\
& + 2(j_1^2 - aj)\eta^3\zeta + (ab - kk_1)\zeta^3\xi + (3ck_1 - 2i_1l - ij)\xi^3\eta \\
& + 2(k^2 - ak_1)\eta\zeta^3 + (3bj - 2il - i_1k_1)\zeta\xi^3 + (ca - jj_1)\xi\eta^3 \\
& + 4(al - j_1k)\eta^2\zeta^2 + (ki + 2k_1l - 3bj_1)\zeta^2\xi^2 \\
& \qquad + (i_1j_1 + 2lj - 3ck)\xi^2\eta^2 \\
& + (4l^2 + 2i_1k + 2ij_1 - 8jk_1)\xi^2\eta\zeta + (7kj - 6j_1l - ai_1)\xi\eta^2\zeta \\
& \qquad + (7k_1j_1 - 6kl - ai)\xi\eta\zeta^2. \\
\mathrm{g} = {} & 0\xi^4 + (jj_1 - ca)\eta^4 + 0\zeta^4 \\
& + (3ai_1 - 2j_1l - jk)\eta^3\zeta + 2(k_1^2 - bk)\zeta^3\xi + (bc - ii_1)\xi^3\eta \\
& + (ab - kk_1)\eta\zeta^3 + 2(i^2 - bi_1)\zeta\xi^3 + (3ck - 2jl - j_1i_1)\xi\eta^3 \\
& + (j_1k_1 + 2lk - 3ai)\eta^2\zeta^2 + 4(bl - k_1i)\zeta^2\xi^2 \\
& \qquad + (ij + 2i_1l - 3ck_1)\eta^2\xi^2 \\
& + (7i_1k_1 - 6il - bj)\xi^2\eta\zeta + (4l^2 + 2j_1i + 2jk_1 - 8ki_1)\xi\eta^2\zeta \\
& \qquad + (7ik - 6k_1l - bj_1)\xi\eta\zeta^2.
\end{aligned}\right\} \quad \ldots\ldots (13).$$

$$
\left.\begin{aligned}
\mathrm{h} = {} & 0\xi^4 + 0\eta^4 + (kk_1 - ab)\,\zeta^4 \\
& + (ca - jj_1)\,\eta^3\zeta + (3bj_1 - 2k_1 l - ki)\,\zeta^3\xi + 2\,(i_1^2 - ci)\,\xi^3\eta \\
& + (3ai - 2kl - k_1 j_1)\,\eta\zeta^3 + (bc - ii_1)\,\zeta\xi^3 + 2\,(j^2 - cj_1)\,\xi\eta^3 \\
& + (jk + 2j_1 l - 3ai_1)\,\eta^2\zeta^2 + (k_1 i_1 + 2li - 3bj)\,\zeta^2\xi^2 + 4\,(cl - i_1 j)\,\xi^2\eta^2 \\
& + (7ji - 6i_1 l - ck_1)\,\xi^2\eta\zeta + (7j_1 i_1 - 6jl - ck)\,\xi\eta^2\zeta \\
& \qquad + (4l^2 + 2k_1 j + 2ki_1 - 8ij_1)\,\xi\eta\zeta^2 .
\end{aligned}\right\} \quad \ldots . (13).
$$

Substituting these values, the result after all reductions becomes

$$0 = \mathbf{F}U = \ldots\ldots\ldots\ldots\ldots \quad (14),$$

$$
\begin{aligned}
& \xi^6\,(6bcii_1 - 4i^3c - 4i_1^3b + 3i^2i_1^2 - b^2c^2) \\
& + \eta^6\,(6cajj_1 - 4j^3a - 4j_1^3c + 3j^2j_1^2 - c^2a^2) \\
& + \zeta^6\,(6abkk_1 - 4k^3b - 4k_1^3a + 3k^2k_1^2 - a^2b^2) \\
& + \eta^5\zeta\,(5ca^2i_1 - 17jj_1ai_1 + 24aj^2l + 12j_1^3i_1 - 5cajk \\
& \qquad + 12cj_1^2k - 7j^2j_1k - 12caj_1l - 12j_1^2jl) \\
& + \zeta^5\xi\,(5ab^2j_1 - 17kk_1bj_1 + 24bk^2l + 12k_1^3j_1 - 5abki \\
& \qquad + 12ak_1^2i - 7k^2k_1i - 12abk_1l - 12k_1^2kl) \\
& + \xi^5\eta\,(5bc^2k_1 - 17ii_1ck_1 + 24ci^2l + 12i_1^3k_1 - 5bcij \\
& \qquad + 12bi_1^2j - 7i^2i_1j - 12bci_1l - 12i_1^2il) \\
& + \eta\zeta^5\,(5a^2bi - 17kk_1ai + 24ak_1^2l + 12k^3i - 5baj_1k_1 \\
& \qquad + 12bk^2j_1 - 7kk_1^2j_1 - 12bakl - 12k^2k_1l) \\
& + \zeta\xi^5\,(5b^2cj - 17ii_1bj + 24bi_1^2l + 12i^3j - 5cbk_1i_1 \\
& \qquad + 12ci^2k_1 - 7ii_1^2k_1 - 12cbil - 12i^2i_1l) \\
& + \xi\eta^5\,(5c^2ak - 17jj_1ck + 24cj_1^2l + 12j^3k - 5aci_1j_1 \\
& \qquad + 12aj^2i_1 - 7jj_1^2i_1 - 12acjl - 12j^2j_1l) \\
& + \eta^4\zeta^2\,(-5ca^2i + 15jj_1ai - 46ajl^2 - 10j_1^3i + 12cakl - 12cj_1k^2 + 4j^2k^2 \\
& \qquad + 5caj_1k_1 + 5jj_1^2k_1 - 10aj^2k_1 - 34j_1^2ki_1 + 26jj_1kl + 34aj_1i_1l \\
& \qquad - 6a^2i_1^2 + 10j_1^2l^2 + 12ajki_1) \\
& + \zeta^4\xi^2\,(-5ab^2j + 15kk_1bj - 46bkl^2 - 10k_1^3j + 12abil - 12ak_1i^2 \\
& \qquad + 4k^2i^2 + 5abk_1i_1 + 5kk_1^2i_1 - 10bk^2i_1 - 34k_1^2ij_1 + 26kk_1il \\
& \qquad + 34bk_1j_1l - 6b^2j_1^2 + 10k_1^2l^2 + 12bkij_1) \\
& + \xi^4\eta^2\,(-5bc^2k + 15ii_1ck - 46cil^2 - 10i_1^3k + 12bcjl - 12bi_1j^2 + 4i^2j^2 \\
& \qquad + 5bci_1j_1 + 5ii_1^2j_1 - 10ci^2j_1 - 34i_1^2jk_1 + 26ii_1jl + 34ci_1k_1l \\
& \qquad - 6c^2k_1^2 + 10i_1^2l^2 + 12cijk_1) \\
& + \eta^2\zeta^4\,(-5ba^2i_1 + 15kk_1ai_1 - 46ak_1l^2 - 10k^3i_1 + 12baj_1l - 12bkj_1^2 \\
& \qquad + 4k_1^2j_1^2 + 5bakj + 5k_1k^2i - 10ak_1^2j - 34k^2j_1i + 26kk_1j_1l \\
& \qquad + 34akil - 6a^2i^2 + 10k^2l^2 + 12ak_1j_1i)
\end{aligned}
$$

$$\begin{aligned}
+ \zeta^2\xi^4 (&- 5cb^2j_1 + 15ii_1bj_1 - 46bi_1l^2 - 10i^3j_1 + 12cbk_1l - 12cik_1^2 \\
&+ 4i_1^2k_1^2 + 5cbik + 5i_1i^2j - 10bi_1^2k - 34i^2k_1j + 26ii_1k_1l \\
&+ 34bijl - 6b^2j^2 + 10i^2l^2 + 12bi_1k_1j)
\end{aligned}$$

$$\begin{aligned}
+ \xi^2\eta^4 (&- 5ac^2k_1 + 15jj_1ck_1 - 46cj_1l^2 - 10j^3k_1 + 12aci_1l - 12aji_1^2 \\
&+ 4j_1^2i_1^2 + 5acji + 5j_1j^2k - 10cj_1^2i - 34j^2i_1k + 26jj_1i_1l \\
&+ 34cjkl - 6c^2k^2 + 10j^2l^2 + 12cj_1i_1k)
\end{aligned}$$

$$\begin{aligned}
+ \eta^3\zeta^3 (&32j_1i_1k^2 + 32j_1^2ki - 20j_1kl^2 - 10jj_1kk_1 - 30ali_1k - 30alij_1 \\
&+ 28al^3 + 44aljk_1 - 14aijk - 14ai_1j_1k_1 - 6bajj_1 - 6cakk_1 \\
&+ 2a^2bc + 4bj_1^3 + 4ck^3 - 14jk^2l - 14j_1^2k_1l + 12a^2ii_1)
\end{aligned}$$

$$\begin{aligned}
+ \zeta^3\xi^3 (&32k_1j_1i^2 + 32k_1^2ij - 20k_1il^2 - 10kk_1ii_1 - 30blj_1i - 30bljk_1 \\
&+ 28bl^3 + 44blki_1 - 14bijk - 14bi_1j_1k_1 - 6cbkk_1 - 6abii_1 \\
&+ 2ab^2c + 4ck_1^3 + 4ai^3 - 14ki^2l - 14k_1^2i_1l + 12b^2jj_1)
\end{aligned}$$

$$\begin{aligned}
+ \xi^3\eta^3 (&32i_1k_1j^2 + 32i_1^2jk - 20i_1jl^2 - 10ii_1jj_1 - 30clk_1j - 30clki_1 \\
&+ 28cl^3 + 44clij_1 - 14cijk - 14ci_1j_1k_1 - 6acii_1 - 6bcjj_1 \\
&+ 2abc^2 + 4ai_1^3 + 4bj^3 - 14ij^2l - 14i_1^2j_1l + 12c^2kk_1
\end{aligned}$$

$$\begin{aligned}
+ \xi^4\eta\zeta (&65ii_1k_1j + 49ii_1l^2 + 11ii_1^2k + 11i^2i_1j_1 - 21cbjk_1 + 23bcl^2 \\
&+ 9bcki_1 + 9bcij_1 - 14cik_1l - 14bi_1jl + 14bij^2 + 14ci_1k_1^2 \\
&- 58i^2jl - 58i_1^2k_1l - 20bi_1^2j_1 - 20ci^2k)
\end{aligned}$$

$$\begin{aligned}
+ \eta^4\zeta\xi (&65jj_1i_1k + 49jj_1l^2 + 11jj_1^2i + 11j^2j_1k_1 - 21acki_1 + 23cal^2 \\
&+ 9caij_1 + 9cajk_1 - 14aji_1l - 14cj_1kl + 14cjk^2 + 14cj_1i^2 \\
&- 58j^2kl - 58j_1^2i_1l - 20cj_1^2k_1 - 20aj^2i)
\end{aligned}$$

$$\begin{aligned}
+ \zeta^4\eta\xi (&65kk_1j_1i + 49kk_1l^2 + 11kk_1^2j + 11k^2k_1i_1 - 21baij_1 + 23abl^2 \\
&+ 9abjk_1 + 9abki_1 - 14bkj_1l - 14ak_1il + 14aki^2 + 14ak_1j^2 \\
&- 58k^2il - 58k_1^2j_1l - 20ak_1^2i_1 - 20bk^2j)
\end{aligned}$$

$$\begin{aligned}
+ \xi^3\eta^2\zeta (&19bcjk - 32bcj_1l - 5abci_1 - 70ii_1jk - 60ii_1j_1l - 5aii_1^2 \\
&+ 50cikl - 7cij_1k_1 + 10aci^2 + 38i_1^2kl + 56i_1^2k_1j_1 + 21ck_1^2j \\
&- 19ck_1l^2 + 8i_1k_1jl - 42i_1l^3 - 43ij^2k_1 + 85ijl^2 - 37ci_1kk_1 \\
&+ 28bi_1jj_1 - 10bj^2l + 15i^2jj_1)
\end{aligned}$$

$$\begin{aligned}
+ \eta^3\zeta^2\xi (&19caki - 32cak_1l - 5abcj_1 - 70jj_1ki - 60jj_1k_1l - 5bjj_1^2 \\
&+ 50ajil - 7ajk_1i_1 + 10baj^2 + 38j_1^2il + 56j_1^2i_1k_1 + 21ai_1^2k \\
&- 19ai_1l^2 + 8j_1i_1kl - 42j_1l^3 - 43jk^2i_1 + 85jkl^2 - 37aj_1ii_1 \\
&+ 28cj_1kk_1 - 10ck^2l + 15j^2kk_1)
\end{aligned}$$

$$\begin{aligned}
+ \zeta^3\xi^2\eta (&19abij - 32abi_1l - 5abck_1 - 70kk_1ij - 60kk_1i_1l - 5ckk_1^2 \\
&+ 50bkjl - 7bki_1j_1 + 10cbk^2 + 38k_1^2jl + 56k_1^2j_1i_1 + 21bj_1^2i \\
&- 19bj_1l^2 + 8k_1j_1il - 42k_1l^3 - 43ki^2j_1 + 85kil^2 - 37bk_1jj_1 \\
&+ 28ak_1ii_1 - 10ai^2l + 15k^2ii_1)
\end{aligned}$$

$$
\begin{aligned}
&+ \xi^3\eta\zeta^2 (19bcj_1k_1 - 32bckl - 5abci - 70ii_1j_1k_1 - 60ii_1kl - 5ai_1i^2 \\
&\qquad + 50bi_1j_1l - 7bi_1kj + 10abi_1^2 + 38i^2j_1l + 56i^2kj + 21bj^2k_1 \\
&\qquad - 19bjl^2 + 8ijk_1l - 42il^3 - 43i_1k_1^2j + 85i_1k_1l^2 - 37bijj_1 \\
&\qquad + 28cikk_1 - 10ck_1^2l + 15i_1^2kk_1) \\
&+ \eta^3\zeta\xi^2 (19cak_1i_1 - 32cail - 5abcj - 70jj_1k_1i_1 - 60jj_1il - 5bj_1j^2 \\
&\qquad + 50cj_1k_1l - 7cj_1ik + 10bcj_1^2 + 38j^2k_1l + 56j^2ik + 21ck^2i_1 \\
&\qquad - 19ckl^2 + 8jki_1l - 42jl^3 - 43j_1i_1^2k + 85j_1i_1l^2 - 37cjkk_1 \\
&\qquad + 28ajii_1 - 10ai_1^2l + 15j_1^2ii_1) \\
&+ \zeta^3\xi\eta^2 (19abi_1j_1 - 32abjl - 5abck - 70kk_1i_1j_1 - 60kk_1jl - 5ck_1k^2 \\
&\qquad + 50ak_1i_1l - 7ak_1ji + 10cak_1^2 + 38k^2i_1l + 56k^2ji + 21ai^2j_1 \\
&\qquad - 19ail^2 + 8kij_1l - 42kl^3 - 43k_1j_1^2i + 85k_1j_1l^2 - 37akii_1 \\
&\qquad + 28bkjj_1 - 10bj_1^2l + 15k_1^2jj_1), \\
&+ \xi^2\eta^2\zeta^2 (84ii_1j_1k + 84jj_1k_1i + 84kk_1i_1j - 12j^2k_1^2 - 12k^2i_1^2 - 12i^2j_1^2 \\
&\qquad - 90ijkl - 90i_1j_1k_1l + 36aii_1l + 36bjj_1l + 36ckk_1l - 16l^2jk_1 \\
&\qquad - 16l^2ki_1 - 16l^2ij_1 + 36l^4 + 24abcl - 8abji_1 - 8bckj_1 \\
&\qquad - 8caik_1 - 22aji^2 - 22bkj^2 - 22cik^2 - 22ai_1^2k_1 - 22bj_1^2i_1 \\
&\qquad - 22ck_1^2j_1).
\end{aligned}
$$

It would be desirable, in conjunction with the above, to obtain the equation

$$K(U) = 0,$$

which results from the elimination of x, y, z from the equations

$$\frac{dU}{dx} = 0, \quad \frac{dU}{dy} = 0, \quad \frac{dU}{dz} = 0, \quad \ldots\ldots \text{ (15)},$$

(*i.e.* the condition of a curve of the third order having a multiple point), but to effect this would be exceedingly laborious. The following is the process of the elimination as given by Dr. Hesse, *Crelle*, t. XXVIII. (and which applies also to the case of any three equations of the second order). Forming the function ∇U, of the third order in x, y, z, by means of the equation

$$\nabla U = \begin{vmatrix} L, & T, & S \\ T, & M, & R \\ S, & R, & N \end{vmatrix} \quad \ldots\ldots\ldots\ldots \text{ (16)},$$

(L, M, N, R, S, T, the same as before).

Then, in consequence of the equations (15), we have not only

$$\nabla U = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(17)},$$

which is very easily proved to be the case, but also

$$\frac{d}{dx}\nabla U = 0, \quad \frac{d}{dy}\nabla U = 0, \quad \frac{d}{dz}\nabla U = 0 \ \ldots \text{ (18)},$$

as will be shown in a subsequent paper "On Points of Inflection."

And from the six equations (15), (18), the six quantities $x^2, y^2, z^2, yz, zx, xy$, may be linearly eliminated, we have

$$\nabla U = Ax^3 + By^3 + Cz^3 + 3Iy^2z + 3Jz^2x + 3Kx^2y + 3I_1yz^2 + 3J_1zx^2 + 3K_1xy^2 + 6\Lambda xyz \;\ldots\; (19),$$

where

$$\left.\begin{aligned}
A &= ak_1j + 2kj_1l - al^2 - jk^2 - j_1^2k_1,\\
B &= bi_1k + 2ik_1l - bl^2 - ki^2 - k_1^2i_1,\\
C &= cj_1i + 2ji_1l - cl^2 - ij^2 - i_1^2j_1,\\
3I &= bck + bi_1j_1 - ck_1^2 + 2jik_1 - 2bjl + il^2 - i^2j_1 - ii_1k,\\
3J &= cai + cj_1k_1 - ai_1^2 + 2kji_1 - 2ckl + jl^2 - j^2k_1 - jj_1i,\\
3K &= abj + ak_1i_1 - bj_1^2 + 2ikj_1 - 2ail + kl^2 - k^2i_1 - kk_1j,\\
3I_1 &= bcj_1 + cik - bj^2 + 2k_1i_1j - 2ck_1l + i_1l^2 - i_1^2k - ii_1j_1,\\
3J_1 &= cak_1 + aji - ck^2 + 2i_1j_1k - 2ai_1l + j_1l^2 - j_1^2i - jj_1k_1,\\
3K_1 &= abi_1 + bkj - ai^2 + 2j_1k_1i - 2bi_1l + k_1l^2 - k_1^2j - kk_1i_1,\\
6\Lambda &= abc + 3ijk + 3i_1j_1k_1 + 2l^3 - aii_1 - bjj_1 - ckk_1 - 2ljk_1\\
&\quad - 2lki_1 - 2lij_1.
\end{aligned}\right\}\ldots(20),$$

And the result of the elimination is $\ldots\ldots(21)$

$$K(U) = \begin{vmatrix} a, & k_1, & j, & l, & j_1, & k, \\ k, & b, & i_1, & i, & l, & k_1, \\ j_1, & i, & c, & i_1, & j, & l, \\ A, & K_1, & J, & L, & J_1, & K, \\ K, & B, & I_1, & I, & L, & K_1, \\ J_1, & I, & C, & I_1, & J, & L, \end{vmatrix} = 0.$$

[$K(U)$ is consequently, as is well known, a function of the twelfth order in $a, b, c, i, j, k, i_1, j_1, k_1, l$].

The equation

$$\nabla U = 0,$$

combined with that of the curve, determine, as Dr. Hesse has demonstrated in the paper quoted, the points of inflection of the curve. It may be inferred from this, that if U reduce itself to the form

$$U = (\alpha x^2 + \beta y^2 + \gamma z^2 + 2\iota yz + 2\kappa xz + 2\lambda xy)\,P = VP \ldots (22),$$

P a linear function of x, y, z: then ∇U takes the form

$$\nabla U = P(\rho V + \sigma P^2) \ldots\ldots\ldots\ldots (23),$$

where ρ is of the second order in the coefficients of P, and also in the coefficients $\alpha, \beta, \gamma, \iota, \kappa, \lambda$: and σ is equal to the determinant

$$\begin{vmatrix} \alpha, & \lambda, & \kappa \\ \lambda, & \beta, & \iota \\ \kappa, & \iota, & \gamma \end{vmatrix} \ldots\ldots\ldots\ldots (24),$$

multiplied by a numerical factor. If U is of the form

$$U = PQR \ldots\ldots\ldots\ldots\ldots (25),$$

then

$$\nabla U = \rho PQR = \rho U \ldots\ldots\ldots\ldots (26).$$

And this equation is consequently the condition of the function U being resolvable into linear factors. The equation in question resolves itself into

$$\frac{A}{a} = \frac{B}{b} = \frac{C}{c} = \frac{I}{i} = \frac{J}{j} = \frac{K}{k} = \frac{I_1}{i_1} = \frac{J_1}{j_1} = \frac{K_1}{k_1} = \frac{\Lambda}{l} \ldots (27);$$

a system which must contain three independent equations only. It would be interesting to verify this *à posteriori.*

ON LINEAR TRANSFORMATIONS.*

By Arthur Cayley.

[*Continued from Vol. IV. p.* 209.]

In continuing my researches on the present subject, I have been led to a new manner of considering the question, which, at the same time that it is much more general, has the advantage of applying directly to the only case which one can possibly hope to develope with any degree of completeness, that of functions of two variables. In fact the question may be proposed, "To find all the derivatives of any number of functions, which have the property of preserving their form unaltered after any linear transformations of the variables." By Derivative I understand a function deduced in any manner whatever from the given functions, and I give the name of Hyperdeterminant Derivative, or simply of Hyperdeterminant, to those derivatives which have the property just enunciated. These derivatives may easily be expressed explicitly, by means of the known method of the separation of symbols. We thus obtain the most general expression of a hyperdeterminant. But there remains a question to be resolved, which appears to present very great difficulties, that of determining the *independent* derivatives, and the relation between these and the remaining ones. I have only succeeded in treating a very particular case of this

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* The present paper was originally written for the *Mathematical Journal;* but it has already been published, together with the one to which it forms a sequel, in *Crelle's Journal,* tom. XXX.

question, which shows however in what way the general problem is to be attacked.

Imagine p series' each of m variables

$$x_1, y_1 \ldots \&c. \quad x_2, y_2 \ldots \&c. \quad x_p, y_p \ldots \&c.,$$

where p is at least as great as m.

Similarly p' series' each of m' variables

$$x_1', y_1' \ldots \&c. \quad x_2', y_2' \ldots, \quad \ldots x'_{p'}, y'_{p'} \ldots \&c.$$

p' at least as great as m', and so on. Let the analogous variables $\dot{x}, \dot{y} \ldots$ be connected with these by the equations

$$\begin{aligned} x &= \lambda\dot{x} + \mu\dot{y} + \ldots \\ y &= \lambda'\dot{x} + \mu'\dot{y} + \ldots \\ &\vdots \\ x' &= \lambda'\dot{x}' + \mu'\dot{y}' + \ldots \\ y' &= \lambda''\dot{x}' + \mu''\dot{y}' + \ldots \\ &\vdots \end{aligned}$$

where $x, y \ldots$ stand for $x_1, y_1 \ldots$ or $x_2, y_2 \ldots$ or $x_p, y_p \ldots$ $x', y' \ldots$ stand for $x_1', y_1' \ldots$ or $x_2', y_2' \ldots$ or $x'_{p'}, y'_{p'}$, &c. . . The coefficients $\lambda, \mu \ldots \lambda', \mu' \ldots$ &c.; $\lambda', \mu' \ldots \lambda'', \mu'' \ldots$ remain the same in all these systems. Suppose, next,

$$\xi = \delta_x, \quad \eta = \delta_y,$$

i.e. $$\xi_1 = \delta_{x_1}, \quad \eta_1 = \delta_{y_1}, \ldots \dot{\xi}_1 = \delta_{\dot{x}_1}, \ldots$$

(where $\delta_x, \delta_y \ldots$ are the symbols of differentiation relative to x, y, &c.). Then evidently

$$\begin{aligned} \dot{\xi} &= \lambda\xi + \lambda'\eta + \ldots \\ \dot{\eta} &= \mu\xi + \mu'\eta + \ldots \\ &\vdots \end{aligned}$$

with similar equations for $\dot{\xi}', \dot{\eta}' \ldots$ Suppose

$$\|\Omega\| = \left\| \begin{matrix} \xi_1, \xi_2 \ldots \xi_p \\ \eta_1, \eta_2 \ldots \eta_p \\ \vdots \end{matrix} \right\| \qquad \|\Omega'\| = \left\| \begin{matrix} \xi_1', \xi_2' \ldots \xi'_{p'} \\ \eta_1', \eta_2' \ldots \eta'_{p'} \\ \vdots \end{matrix} \right\|$$

that is to say $\|\Omega\|$ is the series of determinants formed by choosing any m vertical columns to compose a determinant, and similarly $\|\Omega'\|$, &c. Suppose, besides,

$$E = \left| \begin{matrix} \lambda, \mu \ldots \\ \lambda', \mu' \ldots \\ \vdots \end{matrix} \right|, \qquad E' = \left| \begin{matrix} \lambda', \mu' \ldots \\ \lambda'', \mu'' \ldots \\ \vdots \end{matrix} \right|$$

Then, by the known properties of determinants,

$$\|\dot{\Omega}\| = E\|\Omega\| \quad \|\dot{\Omega}'\| = E'\|\Omega'\| \quad \&c.$$

i.e. the terms on the one side are respectively equal to the terms on the other. Hence if

$$\square = F(\|\Omega\|^f, \ \|\Omega'\|^{f'}, . .)$$

i.e. $\square$ a rational and integral function, homogeneous of the order f in the quantities of the series $\|\Omega\|$, homogeneous of the order f' in the quantities of the series $\|\Omega'\|$, &c., we have immediately

$$\dot{\square} = E^f E'^{f'} . . \square.$$

Or if U be any function whatever of the variables $x, y. . . .$ which is transformed by the linear substitutions above into $\dot{U}$, then

$$\dot{\square}\dot{U} = E^f E'^{f'} . . \square U.$$

Or the function $\square U$

is by the above definition a hyperdeterminant derivative. The symbol $\square$ may be called "symbol of hyperdeterminant derivation," or simply " hyperdeterminant symbol."

Let $A, B. .$ represent the different quantities of the series $\|\Omega\|$,—$A', B'. .$ those of the series' $\|\Omega'\|$, &c.. . Then $\square$ may be reduced to a single term, and we may write

$$\square = A^\alpha . B^\beta . . A'^{\alpha'} . B'^{\beta'} . .$$

Also U may be supposed of the form

$$U = \Theta . \Phi . .$$

where Θ, Φ are functions of the variables of one of the sets $x, y. .$ of one of the sets $x', y'. .$ &c. Thus Θ is of the form

$$F(x_1, y_1 . . x_1', y_1 . .),$$

and so on. The functions Θ, $\Phi. . . .$ may be the same or different. It may be supposed after the differentiations that several of the sets $x, y. . .$ or of the sets $x', y'. . . .$ become identical. In such cases it will always be assumed that the functions Θ,. . into which these sets of variables enter, are similar; so that they become absolutely identical, when the variables they contain are made so. Thus the general expression of a hyperdeterminant is

$$\square U = A^\alpha B^\beta . . A'^{\alpha'} B'^{\beta'} . . \Theta\Phi . .$$

in which, after the differentiations, any number of the sets of variables are made equal. For instance, if all the sets $x, y. .$ and all the sets $x', y'. .$ are made equal, the hyperdeterminant refers to a single function $F(x, y. . x', y'. .)$. In any other case it refers not to a single function but to several.

What precedes, is the general theory: it might perhaps have been made clearer by confining it to a particular case:

and by doing this from the beginning, it will be seen that it presents no real difficulties. Passing at present to some developments, to do this, I neglect entirely the sets x', y'.. and I assume that the number m of variables in each of the sets x, y.. reduces itself to two; so that I consider functions of two variables x, y only. The functions Θ, Φ, &c. reduce themselves to functions $V_1, V_2 \ldots V_p$ of the variables x_1, y_1, or $x_2, y_2 \ldots$ or x_p, y_p. Writing also

$$\xi_1\eta_2 - \xi_2\eta_1 = \overline{12}, \text{ \&c.}$$

the symbols A, B.. reduce themselves to $\overline{12}, \overline{13}$.. Hence for functions of two variables, there results the following still tolerably general form

$$\square\, U = \overline{12}^{\alpha}\, \overline{13}^{\beta}\, \overline{14}^{\gamma} \ldots \overline{23}^{\beta'}\, \overline{24}^{\gamma'} \ldots \overline{34}^{\gamma''} \ldots V_1 V_2 V_3 V_4 \ldots$$

The functions V_1, V_2.. may be the same or different: but they will be supposed the same whenever the corresponding variables are made equal. This equality will be denoted by writing, for instance,

$$\square\, V V' V V \ldots$$

to represent the value assumed by

$$\square\, V_1 V_2 V_3 V_4 \ldots$$

when after the differentiations

$$\begin{aligned} &x_1, y_1 = x_3, y_3 = x_4, y_4 = x, y \\ &x_2, y_2 = x', y'. \\ &\text{\&c.} \end{aligned}$$

It is easy to determine the general term of $\square\, U$. To do this, writing for shortness

$$\begin{aligned} \alpha + \beta + \gamma + \ldots &= f_1 \\ \alpha + \beta' + \gamma' \ldots &= f_2 \\ \beta + \beta' + \gamma'' &= f_3 \\ \text{\&c.} \end{aligned}$$

$$N = (-)^{r+s+t\ldots+r'+s'\ldots+t'\ldots} \frac{[\alpha]^r}{[r]^r} \frac{[\beta]^s}{[s]^s} \frac{[\gamma]^t}{[t]^t} \cdots \frac{[\beta']^{s'}}{[s']^{s'}} \frac{[\gamma']^{t'}}{[t']^{t'}} \cdots \frac{[\gamma'']^{t''}}{[t'']^{t''}} \cdots$$

$$\xi^{f-r}\eta^r V \text{ or } \delta_x^{f-r}\delta_y^{r} V = \overset{f}{V}{}^{,r} \text{ or } V^{,r}.$$

The general term is

$$N\, \overset{f_1}{V_1}{}^{,r+s+t\ldots}\, \overset{f_2}{V_2}{}^{,\alpha-r+s'+t'\ldots}\, \overset{f_3}{V_3}{}^{,\beta-s+\beta'-s'+t'\ldots}$$

where $r, s, t, \ldots s', t' \ldots t'', \ldots$ extend from 0 to $\alpha, \beta, \gamma \ldots \beta', \gamma' \ldots \gamma''$.. respectively. It would be easy to change this general term in a way similar to that which will be employed presently for the particular case of $\square\, V_1 V_2 V_3$.

If several of the functions become identical, and for these some of the letters f are equivalent, it is clear that the derivative $\square U$ refers to a certain number of functions $V_1, V_2 ..$ the same or different, of the variables $x, y; x', y' ..$ and besides that this derivative is homogeneous, of the degrees $\theta_1, \theta_1' ..$ with respect to the differential coefficients of the orders $f_1, f_1' ..$ &c. of V_1, (consequently homogeneous of the order $\theta_1 + \theta_1' + ..$ with respect to these differential coefficients collectively), homogeneous and of the degrees $\theta_2, \theta_2' ....$ with respect to the differential coefficients of the orders $f_2, f_2' ..$ of V_2, (consequently of the order $\theta_2 + \theta_2' ..$ with respect to these collectively), and so on. The degree with respect to all the functions is of course $\theta_1 + \theta_1' .. + \theta_2 + \theta_2' + .. = p$ suppose. In general, only a single function will be considered, and it will be assumed that $\square U$ only contains the differential coefficients of the f^{th} order. In this case, the derivative is said to be of the degree p and of the order f. The most convenient classification is by degrees, rather than by orders.

Commencing with the simplest case, that of functions of the second order (and writing V, W instead of V_1, V_2), we have

$$\square VW = \overline{12}^a VW.$$

(where ξ_1, η_1 apply to V and ξ_2, η_2 to W). This will be constantly represented in the sequel by the notation

$$\overline{12}^a VW = B_a(V, W).$$

Hence, writing $\delta_x^{\,a} V = V^{,0} \quad \delta_x^{\,a-1}\delta_y V = V^{,1} ..$

we have $B_a(V, W) = V^{,0} W^{,a} - \dfrac{[a]'}{[1]'} V^{,1} W^{,a-1} + ..$

And in particular, according as a is odd or even,

$$B_a(V, V) = 0$$

$$\tfrac{1}{2} B_a(V, V) = V^{,0} V^{,a} - \frac{a}{1} V^{,1} V^{,a-1} + ..$$

continued to the term which contains $V^{,\frac{1}{2}a} V^{,\frac{1}{2}a}$, the coefficient of this last term, divided by two.

Thus, for the functions $\frac{1}{2}(ax^2 + 2bxy + cy^2)$, $\frac{1}{24}(ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4)$, &c., if a be made equal to 2, 4, &c. respectively, we have the *constant* derivatives

$$ac - b^2$$
$$ae - 4bd + 3c^2$$
$$ag - 6bf + 15ce - 10d^2$$
$$ai - 8bh + 28cg - 56df + 35e^2.$$
$$\vdots$$

which have all of them the property of remaining unaltered,

à un facteur près, when the variables are transformed by means of $x = \lambda\dot{x} + \mu\dot{y}$, $y = \lambda'\dot{x} + \mu'\dot{y}$. Thus, for instance, if these equations give

$$ax^2 + 2bxy + cy^2 = \dot{a}\dot{x}^2 + 2\dot{b}\dot{x}\dot{y} + \dot{c}\dot{y}^2,$$

then $$\dot{a}\dot{c} - \dot{b}^2 = (\lambda\mu' - \lambda'\mu)^2.(ac - b^2),$$

and so on. This is the general property, which we call to mind for the case of these constant derivatives.

The above functions may be transformed by means of the identical equation

$$B_a(V, W) = \overline{12}^{a-k} B_k(V, W),$$

to make use of which, it is only necessary to remark the general formula

$$\xi_1{}^\lambda \eta_1{}^\mu \xi_2{}^\rho \eta_2{}^\sigma\, B_k(V, W) = B_k(\xi^\lambda \eta^\mu V, \xi^\rho \eta^\sigma W).$$

Thus, if $k = 1$, we obtain for the above series, the new forms

$$\begin{array}{l} ac - b^2 \\ (ae - bd) - 3(bd - c^2) \\ (ag - bf) - 5(bf - ce) + 10(ce - d^2) \\ (ai - bh) - 7(bh - cg) + 21(cg - df) - 35(df - e^2) \\ \&c., \end{array}$$

the law of which is evident. This shows also that these functions may be linearly expressed by means of the series of determinants

$$\begin{Vmatrix} a, b \\ b, c \end{Vmatrix} \quad \begin{Vmatrix} a, b, c \\ b, c, d \end{Vmatrix} \quad \&c.$$

We may also immediately deduce from them the derivatives B which relate to two functions. For example, for functions of the sixth order this is

$$ag' + a'g - 6(bf' + b'f) + 15(ce' + c'e) - 20dd',$$

which has an obvious connection with

$$ag - 6bf + 15ce - 10d^2;$$

and the same is the case for functions of any order.

The following theorem is easily verified; but I am unacquainted with the general theory to which it belongs.

"If U, V are any functions of the second order, and $W = \lambda U + \mu V$; then

$$B_2'[B_2(W, W), B_2(W, W)] = 0$$

(where B_2' relates to λ, μ) is the same that would be obtained by the elimination of x, y between $U = 0$, $V = 0$." (See Note.*)

* Not given with the present paper.

In fact this becomes

$$4(ac - b^2)(a'c' - b'^2) - (ac' + a'c - 2bb')^2 = 0,$$

which is one of the forms under which the result of the elimination of the variables from two quadratic equations may be written. This is a result for which I am indebted to Mr. Boole.

Passing to the third degree, we may consider in particular the derivatives

$$\square\, UVW = \overline{23}^{\alpha}\, \overline{31}^{\alpha}\, \overline{12}^{\alpha}\, UVW = C_\alpha(U, V, W)$$

Writing for shortness

$$A_r = \frac{[\alpha]^r}{[r]^r}, \quad \delta_x^{2\alpha - r} \delta_y^{r} U = U_{,r},$$

we have the general term

$$C_\alpha(U, V, W) = \Sigma\, \{(-)^{r+s+t} A_r A_s A_t U_{,\alpha+t-s} V_{,\alpha+r-t} W_{,\alpha+s-r}\},$$

where r, s, t extend from 0 to α. By changing the suffixes r, s the following more convenient formula

$$C_\alpha(U,V,W) = \Sigma\Sigma\, \{(-)^{\sigma+\rho} U_{,\rho} V_{,\sigma} W_{,3\alpha-\rho-\sigma} \Sigma[(-)^t A_{\rho-t} A_{\sigma+t-\alpha} A_t]\},$$

where t extends from 0 to 2α: ρ, σ, and $3\alpha - \rho - \sigma$ must be positive and not greater than 2α.

In particular, according as α is odd or even,

$$C_\alpha(U,U,U) = 0,$$

$$C_\alpha(U,U,U) = 6\Sigma\Sigma\, \{(-)^{\rho+\sigma} U_{,\rho} U_{,\sigma} U_{,3\alpha-\rho-\sigma} \Sigma[(-)^t A_{\rho-t} A_{\sigma+t-\alpha} A_t]\},$$

in omitting such values of ρ, σ for which $\rho > \sigma$ or $\sigma > 3\alpha - \rho - \sigma$, and dividing by two the terms in which $\rho = \sigma$ or $\sigma = 3\alpha - \rho - \sigma$, and by six the term for which $\rho = \sigma = 3\alpha - \rho - \sigma = \alpha$.

In particular, for functions of the fourth or eighth orders we have the constant derivatives

$$ace - ad^2 - b^2e - c^3 + 2bcd$$

$$aei - 4ibd - 4afh + 3ag^2 + 3ic^2 + 12beh - 8chd - 8bgf - 22ceg$$
$$+ 24cf^2 + 24d^2g - 36def + 15e^3.$$

The first of which is a simple determinant. Thus we have been led to the functions $ae - 4bd + 3c^2$ and $ace - ad^2 - eb^2 - c^3 + 2bcd$, which occur in my "Note sur quelques formules, &c." (*Crelle*, tom. XXIV.), and in the forms which M. Eisenstein has given for the solutions of equations of the four first degrees.

Let U be a function of the order 4α: the derivative C may be expressed by means of the derivatives B.

For, consider the function

$$B_{4\alpha}[U, B_{2\alpha}(V, W)],$$

Paying attention to the signification of B, this may be written

$$\overline{1\theta}^{4\alpha}\,\overline{23}^{2\alpha}\,UVW,$$

where the symbols ξ_θ, η_θ refer to the two systems $x_2, y_2 : x_3, y_3$. Thus it is easily seen that we may write

$$\xi_\theta = \xi_2 + \xi_3,\quad \eta_\theta = \eta_2 + \eta_3,\quad \text{or}\quad \overline{1\theta} = \overline{12} + \overline{13} = \overline{12} - \overline{31},$$

whence the function becomes

$$(\overline{12} - \overline{31})^{4\alpha}\,\overline{23}^{2\alpha}\,UVW.$$

Of which all the terms vanish except

$$\frac{[4\alpha]^{2\alpha}}{[2\alpha]^{2\alpha}}\,\overline{12}^{2\alpha}\,\overline{23}^{2\alpha}\,\overline{31}^{2\alpha}\,UVW.$$

Or putting $\qquad K = \dfrac{[4\alpha]^{2\alpha}}{[2\alpha]^{2\alpha}} = \dfrac{2^{4\alpha}\,1.3\,..(4\alpha-1)}{2.4\,..\,4\alpha}$,

we have $\quad B_{4\alpha}[U, B_{2\alpha}(V, W)] = KC_\alpha(U, V, W)$.

Or in particular

$$B_{4\alpha}[U, B_{2\alpha}(U, U)] = KC_\alpha(U, U, U).$$

Thus for example, neglecting a numerical factor,

$$\begin{aligned}&(ax^2 + 2bxy + cy^2)(cx^2 + 2dxy + ey^2) - (bx^2 + 2cxy + dy^2)^2 \\ &= (ac - b^2)\,x^4 + 2\,(ad - bc)\,x^3y + (ae + 2bd - 3c^2)\,x^2y^2 \\ &\qquad + 2\,(be - cd)\,xy^3 + (ce - d^2)\,y^4.\end{aligned}$$

And then

$$\begin{aligned}&e(ac - b^2) - 4d\,\tfrac{2}{4}(ad - bc) + 6c\,\tfrac{1}{6}(ae + 2bd - 3c^2) \\ &\qquad - 4b\,\tfrac{2}{4}(be - cd) + a(ce - d^2) \\ &= 3\,(ace - ad^2 - be^2 - c^3 + 2bcd).\end{aligned}$$

We have likewise the singular equation

$$B_{2\alpha}(V, W) = K\left(x^{4\alpha}\frac{d}{da_{4\alpha}} - x^{4\alpha-1}y\frac{d}{da_{4\alpha-1}}\,..+y^{4\alpha}\frac{d}{da_0}\right)C_\alpha(U, V, W)$$

where $U = \dfrac{1}{[4\alpha]^{4\alpha}}\left(a_0x^{4\alpha} - \dfrac{[4\alpha]^1}{1}\,a_1x^{4\alpha-1}y\,..+a_{4\alpha}y^{4\alpha}\right)$, &c.

If however $U = V = W$, we must write

$$B_{2\alpha}(U, U) = \tfrac{1}{3}K\left(x^{4\alpha}\frac{d}{da_{4\alpha}} - x^{4\alpha-1}y\frac{d}{da_{4\alpha-1}}\,..+y^{4\alpha}\frac{d}{da_0}\right)C_\alpha(U, U, U),$$

the reason of which is easily seen. This subject will be resumed in the sequel.

The functions C may be transformed in the same way as the functions B have been. In fact

$$C_a(U, V, W) = \overline{12}^{a-k}\,\overline{23}^{a-k}\,\overline{31}^{a-k}\,C_k(U, V, W),$$

if in particular $k = 1$.

$$C_1(U, V, W) = \begin{vmatrix} U_{,0} & U_{,1} & U_{,2} \\ V_{,0} & V_{,1} & V_{,2} \\ W_{,0} & W_{,1} & W_{,2} \end{vmatrix} \qquad U_{,0} \text{ for } \overset{2}{U}_{,0}\ldots$$

But in general

$\xi_1^{\rho'}\eta_1^{\rho}\xi_2^{\sigma'}\eta_2^{\sigma}\xi_3^{\tau'}\eta_3^{\tau}C_1(U,V,W)$, where $\rho+\rho'=\sigma+\sigma'=\tau+\tau'=2a-2$,

$$= C_1(\overset{2a-2}{U}_{,\rho}\overset{2a-2}{V}_{,\sigma}\overset{2a-2}{W}_{,\tau}) = \begin{vmatrix} U_{,\rho} & U_{,\rho+1} & U_{,\rho+2} \\ V_{,\sigma} & V_{,\sigma+1} & V_{,\rho+2} \\ W_{,\tau} & W_{,\tau+1} & W_{,\rho+2} \end{vmatrix} \qquad U_{,\rho} \text{ for } \overset{2a}{U}_{,\rho}, \text{ \&c.}$$

whence $C_a(U,V,W)$

$$= \Sigma\Sigma\left\{(-)^{\rho+\sigma} \begin{vmatrix} U_{,\rho} & V_{,\sigma-1} & W_{,3a-\rho-\sigma-2} \\ U_{,\rho+1} & V_{,\sigma} & W_{,3a-\rho-\sigma-1} \\ U_{,\rho+2} & V_{,\sigma+1} & W_{,3a-\rho-\sigma} \end{vmatrix} \Sigma[(-)^t A_t' A'_{\rho-t} A'_{\sigma-a+t}]\right\},$$

where $A_t' = \dfrac{[a-1]^t}{[t]^t}$; t extends from 0 to $\overline{a-1}$; ρ, $\sigma - 1$, and $3a - \rho - \sigma - 2$ may have any positive values not greater than $2a - 2$.

In particular $C_a(U, U, U)$

$$= 6\Sigma\Sigma\left\{(-)^{\rho+\sigma} \begin{vmatrix} U_{,\rho} & U_{,\sigma-1} & U_{,3a-\rho-\sigma-2} \\ U_{,\rho+1} & U_{,\sigma} & U_{,3a-\rho-\sigma-1} \\ U_{,\rho+2} & U_{,\sigma+1} & U_{,3a-\rho-\sigma} \end{vmatrix} \Sigma[(-)^t A'_t A'_{\rho-t} A'_{\sigma-a+t}]\right\},$$

where ρ, σ need only have such values that $\rho < \sigma - 1$, $\sigma - 1 < 3a - \rho - \sigma - 2$.

In particular the derivative $aei - \ldots + 15e^3$ may be transformed into

$$\begin{vmatrix} a, d, g \\ b, e, h \\ c, f, i \end{vmatrix} - 3\begin{vmatrix} a, e, f \\ b, f, g \\ c, g, h \end{vmatrix} - 3\begin{vmatrix} b, c, g \\ c, d, h \\ d, e, i \end{vmatrix} + 6\begin{vmatrix} b, d, f \\ c, e, g \\ d, f, h \end{vmatrix} - 15\begin{vmatrix} c, d, e \\ d, e, f \\ e, f, g \end{vmatrix}$$

in which form it is obviously a linear function of the determinants

$$\begin{Vmatrix} a, b, c, d, e, f, g \\ b, c, d, e, f, g, h \\ c, d, e, f, g, h, i \end{Vmatrix}$$

which is true generally.

Omitting for the present the theory of derivatives of the form

$$\square\, UVW = \overline{23}^{\alpha}\,\overline{31}^{\beta}\,\overline{12}^{\gamma}\, UVW,$$

we pass on to the derivatives of the fourth degree, considering those forms in which all the differential coefficients are of the same order. We may write

$$\square\, UVWX = (\overline{12}.\overline{34})^{\alpha}\,(\overline{13}.\overline{42})^{\beta}\,(\overline{14}.\overline{23})^{\gamma}\, UVWX$$
$$= D_{\alpha,\beta,\gamma}(U, V, W, X) = D_{\alpha,\beta,\gamma};$$

or if, for shortness,

$$\overline{12}.\overline{34} = \mathfrak{A}, \quad \overline{13}.\overline{42} = \mathfrak{B}, \quad \overline{14}.\overline{23} = \mathfrak{C},$$

we have $\qquad D_{\alpha,\beta,\gamma} = \mathfrak{A}^{\alpha}\,\mathfrak{B}^{\beta}\,\mathfrak{C}^{\gamma}.UVWX.$

Suppose $U = V = W = X$, and consider the derivatives which correspond to the same value f of $\alpha + \beta + \gamma$. The question is to determine how many of these are independent, and to express the remaining ones in terms of these. Since the functions become equal after the differentiations, we are at liberty before the differentiations to interchange the symbolic number 1, 2, 3, 4 in any manner whatever. We have thus

$$D_{\alpha,\beta,\gamma} = D_{\beta,\gamma,\alpha} = D_{\gamma,\alpha,\beta} = (-)^f D_{\alpha,\gamma,\beta} = (-)^f D_{\gamma,\beta,\alpha} = (-)^f D_{\beta,\alpha,\gamma}.$$

But the identical equation

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} = 0:$$

multiplied this by $\mathfrak{A}^a\,\mathfrak{B}^b\,\mathfrak{C}^c$ and applied to each term to the product $UVWX$, gives

$$D_{a+1,b,c} + D_{a,b+1,c} + D_{a,b,c+1} = 0;$$

whence if $a + b + c = f - 1$, we have a set of equations between the derivatives $D_{\alpha,\beta,\gamma}$ for which $\alpha + \beta + \gamma = f$. Reducing these by the conditions first found, suppose Θf is the number of divisions of an integer f into three parts, zero admissible, but permutations of the same three parts rejected. The number of derivatives is Θf, and the number of relations between them is $\Theta(f-1)$. Hence $\Theta f - \Theta(f-1)$ of these derivatives are independent: only when f is even, one of these is $D_{f,0,0}$, *i.e.* $\overline{12}^f\,\overline{34}^f.UVWX$, *i.e.* $\overline{12}^f UV.\overline{34}^f WX$, or $B_f(U, V)\,B_f(X, W)$, *i.e.* $[B_f(U, U)]^2$. Rejecting this, the number of independent derivatives when f is even, is $\Theta f - \Theta(f-1) - 1$. Let $E\left(\dfrac{a}{b}\right)$ be the greatest integer contained in the fraction $\dfrac{a}{b}$; the number required may be shown to be

$$E\frac{f}{6} \text{ or } E\frac{f+3}{6},$$

according as f is even or odd. Giving to f the six forms

$$6g,\quad 6g+1,\quad 6g+2,\quad 6g+3,\quad 6g+4,\quad 6g+5,$$

the corresponding numbers of the independent derivatives are

$$g,\quad g,\quad g,\quad g+1,\quad g,\quad g+1.$$

Thus there is a single derivative for the orders 3, 5, 6, 7, 8, 10 .... two for the orders 9, 11, 12, 13, 14, 16....&c.

When f is even, the terms $D_{f-3\cdot 3\cdot 0}$, $D_{f-6\cdot 6\cdot 0}$ .. and when f is odd, the terms $D_{f-1\cdot 1\cdot 0}$, $D_{f-4\cdot 4\cdot 0}$, $D_{f-7\cdot 7\cdot 0}$, &c. may be taken for independent derivatives: by stopping immediately before that in which the second suffix exceeds the first, the right number of terms is always obtained. Thus, when $f=9$ the independent derivatives are $D_{8.1.0}$, $D_{5.4.0}$, and we have the system of equations

$$\begin{array}{ll}
D_{900}+D_{810}+D_{801}=0, & D_{621}+D_{531}+D_{522}=0,\\
D_{810}+D_{720}+D_{711}=0, & D_{540}+D_{450}+D_{441}=0,\\
D_{720}+D_{630}+D_{621}=0, & D_{531}+D_{441}+D_{432}=0,\\
D_{711}+D_{621}+D_{612}=0, & D_{522}+D_{432}+D_{423}=0,\\
D_{630}+D_{531}+D_{540}=0, & D_{432}+D_{342}+D_{333}=0,
\end{array}$$

which are to be reduced by

$$D_{900}=-D_{900}=0,\qquad D_{801}=-D_{810},\ \&c.$$

It is easy to form the table

$D_{200}=B_2^2$	$D_{500}=0$	$D_{700}=0$
$D_{110}=-\frac{1}{2}B_2^2$	D_{410},	D_{610}
	$D_{320}=-D_{410}$	$D_{520}=-D_{610}$
$D_{300}=0$	$D_{311}=0$	$D_{511}=0$
D_{210}	$D_{221}=0$	$D_{430}=D_{610}$
$D_{111}=0$		$D_{421}=0$
	$D_{600}=B_6^2$	$D_{331}=0$
$D_{400}=B_4^2$	$D_{510}=-\frac{1}{2}B_6^2$	$D_{222}=0$
$D_{310}=-\frac{1}{2}B_4^2$	$D_{420}=-\frac{2}{3}D_{330}+\frac{1}{6}B_6^2$	
$D_{220}=\frac{1}{2}B_4^2$	$D_{411}=\frac{2}{3}D_{330}+\frac{1}{3}B_6^2$	
$D_{211}=0$	D_{330},	
	$D_{321}=-\frac{1}{3}D_{330}-\frac{1}{6}B_6^2$	
	$D_{222}=\frac{2}{3}D_{330}+\frac{1}{3}B_6^2$	

$$\begin{array}{ll}
D_{800} = B_8^2 & D_{900} = 0 \\
D_{710} = -\tfrac{1}{2} B_8^2 & D_{810}\,, \\
D_{620} = -\tfrac{2}{3} D_{530} + \tfrac{1}{6} B_8^2 & D_{720} = -D_{810} \\
D_{611} = \tfrac{2}{3} D_{530} + \tfrac{1}{3} B_8^2 & D_{711} = 0 \\
D_{530}\,, & D_{630} = \tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540} \\
D_{521} = -\tfrac{1}{3} D_{530} - \tfrac{1}{12} B_8^2 & D_{621} = \tfrac{1}{2} D_{810} + \tfrac{1}{2} D_{540} \\
D_{440} = -\tfrac{16}{15} D_{530} - \tfrac{1}{30} B_8^2 & D_{540}\,, \\
D_{431} = \tfrac{1}{15} D_{530} - \tfrac{1}{30} B_8^2 & D_{531} = -\tfrac{1}{2} D_{810} - \tfrac{1}{2} D_{540} \\
D_{422} = \tfrac{4}{15} D_{530} + \tfrac{2}{15} B_8^2 & D_{522} = 0 \\
D_{332} = -\tfrac{2}{15} D_{530} - \tfrac{1}{15} B_8^2 & D_{441} = 0 \\
 & D_{432} = \tfrac{1}{2} D_{810} + \tfrac{1}{2} D_{540} \\
 & D_{333} = 0.
\end{array}$$

Whatever be the value, all the tables except the three first commence thus, according as f is even or odd,

$$\begin{array}{ll}
D_{f,0,0} = B_f^2 & \text{or } D_{f,0,0} = 0 \\
D_{f-1,1,0} = -\tfrac{1}{2} B_f^2 & D_{f-1,1,0} \\
D_{f-2,2,0} = -\tfrac{2}{3} D_{f-3,3,0} + \tfrac{1}{6} B_f^2 & D_{f-2,2,0} = -D_{f-1,1,0} \\
D_{f-2,1,1} = \tfrac{2}{3} D_{f-3,3,0} + \tfrac{1}{3} B_f^2 & D_{f-2,1,1} = 0 \\
D_{f-3,3,0} & \vdots \\
\vdots &
\end{array}$$

but beyond this I am not acquainted with the law.

To give some formulæ for the transformation of these derivatives; we have, for example,

$$D_{f-1,1,0} = (\overline{12}\,.\,\overline{34})^{f-1}\,\overline{13}\,.\,\overline{42}\; UUUU$$
$$= \overline{13}\,.\,\overline{42}\; B_{f-1}(U, U)\, B_{f-1}(U, U).$$

But $\quad \overline{12}\,.\,\overline{42} = \xi_1\eta_2\eta_3\xi_4 - \xi_1\xi_2\eta_3\eta_4 - \eta_1\eta_2\xi_3\xi_4 + \eta_1\xi_2\xi_3\eta_4,$

and $\quad \xi_1\eta_2\eta_3\xi_4\, B_{f-1}(U,U)\, B_{f-1}(U,U)$

$$= B_{f-1}(\xi U, \eta U)\, B_{f-1}(\eta U, \xi U)$$
$$= B_{f-1}(U^{,0} U^{,1})\, B_{f-1}(U^{,1} U^{,0}), \text{ \&c.}$$

(where $U^{,0}$, $U^{,1}$ stand for $\overset{1}{U}{}^{,0}\overset{1}{U}{}^{,1}$), &c.; or

$$D_{f-1,1,0} = -2\,\{B_{f-1}(U^{,0} U^{,0})\, B_{f-1}(U^{,1} U^{,1})$$
$$- B_{f-1}(U^{,0} U^{,1})\, B_{f-1}(U^{,1} U^{,0})\},$$

which reduces itself to

$$D_{f-1,1,0} = -2\,\{B_{f-1}(U^{,0} U^{,1})\}^2,$$
$$D_{f-1,1,0} = -2\,\{B_{f-1}(U^{,0} U^{,0})\, B_{f-1}(U^{,1} U^{,1}) - [B_{f-1}(U^{,0} U^{,1})]^2\},$$

according as f is even or odd.

For example, for the orders 3, 5, 7, 9, we have

$$D_{210} = -2\left\{4(ac-b^2)(bd-c^2)-(ad-bc)^2\right\},$$

$$D_{410} = -2\left\{4(ae-4bd+3c^2)(bf-4ce+3d^2)-(af-3be+2cd)^2\right\}$$

$$D_{610} = -2\left\{4(ag-6bf+15ce-10d^2)(bh-6cg+15df-10e^2) \right.$$
$$\left. -(ah-5bg+9cf-5de)^2\right\}.$$

$$D_{810} = -2\left\{4(ai-8bh+28cg-56df+35e^2)(bj-8ci \right.$$
$$\left. +28dh-56eg+35f^2)-(aj-7bi+20ch-28dg+14ef)^2\right\}.$$

The derivatives D will be presently calculated in a completely expanded form up to the ninth order. We have, therefore, still to find the derivatives of the sixth and eighth orders and a second derivative of the ninth order. For the sixth order, the simplest method is to make use of D_{222}, which is easily seen to be equal to

$$24\begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix}$$

For the two others we have the general formulæ

$$D_{f-2,2,0}$$
$$= 2\left\{B_{f-2}(U^{,0}U^{,0})\,B_{f-2}(U^{,2}U^{,2}) - 4B_{f-2}(U^{,0}U^{,1})\,B_{f-2}(U^{,2}U^{,1}) \right.$$
$$\left. + B_{f-2}(U^{,0}U^{,2})\,B_{f-2}(U^{,2}U^{,0}) + 2\left[B_{f-2}(U^{,1}U^{,1})\right]^2\right\},$$

where $U^{,0}$, $U^{,1}$, $U^{,2}$ have been written for $\overset{2}{U}{}^{,0}$, $\overset{2}{U}{}^{,1}$, $\overset{2}{U}{}^{,2}$; a formula which is demonstrated in precisely the same way as that for $D_{f-1,1,0}$.

$$D_{f-3,3,0}$$
$$= -2\left\{B_{f-3}(U^{,0}U^{,3})\,B_{f-3}(U^{,3}U^{,0}) - 6B_{f-3}(U^{,0}U^{,1})\,B_{f-3}(U^{,3}U^{,2}) \right.$$
$$+ 6B_{f-3}(U^{,0}U^{,2})\,B_{f-3}(U^{,3}U^{,1}) + 9B_{f-3}(U^{,1}U^{,1})\,B_{f-3}(U^{,2}U^{,2})$$
$$\left. - 9B_{f-3}(U^{,1}U^{,2})\,B_{f-3}(U^{,2}U^{,1}) - B_{f-3}(U^{,0}U^{,3})\,B_{f-3}(U^{,3}U^{,0})\right\},$$

(in which $U^{,0}$, &c. stand for $\overset{0}{U}{}^{,0}$, &c.). In particular

$$D_{620} = 2\left\{4(ag-6bf+15ce-10d^2)(ci-6dh+15eg-10f^2) \right.$$
$$-4(ah-5bg+9cf-5de)(bi+5ch+9dg-5ef)$$
$$+(ai-6bh+16cg-26df+15e^2)^2+8(bh-6cg$$
$$\left. +15df-10e^2)^2\right\},$$

$$
\begin{aligned}
D_{630} = -2\{&4(ag - 6bf + 15ce - 10d^2)(dj - 6ei + 15fh - 10g^2) \\
&- 6(ah - 5bg + 9cf - 5de)(cj - 5di + 9eh - 5fg) \\
&+ 6(ai - 6bh + 16cg - 26df + 15e^2)(bj - 6ci + 16dh \\
&- 26eg + 15f^2) + 36(bh - 6cg + 15df - 10e^2)(ci - 6dh \\
&+ 15eg - 10f^2) - 9(bi - 5ch + 9dg - 5ef)^2 - (aj - 6bi \\
&+ 15ch - 19dg + 9ef)^2\}.
\end{aligned}
$$

Hence we have all the elements necessary for the calculation of the following table of the independent constant derivatives of the fourth degree, up to the ninth order.

$$
\begin{aligned}
D_{210} &= -2(6abcd - 4ac^3 - 4b^3d + 3b^2c^2 - a^2d^2), \\
D_{410} &= -2(10aebf - 16ae^2c - 16b^2df + 12aed^2 + 12c^2bf - 48c^3e \\
&\quad - 48d^3b + 76bcde + 32c^2d - a^2f^2 - 4acdf - 9b^2e^2), \\
{}^*D_{222} &= 24(aceg + 2adef + 2gdbc - agd^2 - ae^3 - gc^3 - acf^2 - geb^2 \\
&\quad - 2bd^2f - 2bcef + bde^2 + fdc^2 + b^2f^2 + e^2c^2 - 3ecd^2 \\
&\quad + bde^2 + fdc^2 + d^4), \\
D_{610} &= -2(14agbh + 234bgcf + 990cedf - 375d^2e^2 - a^2h^2 \\
&\quad - 25b^2g^2 - 81c^2f^2 - 18ahcf + 10ahde - 50bgde - 24acy \\
&\quad - 24b^2fh + 60agdf + 60cebh - 40age^2 - 40d^2bh \\
&\quad - 360bdf^2 - 36c^2eg + 240bfe^2 + 240d^2cg - 600ce^3 \\
&\quad - 600d^2bf), \\
{}^\dagger D_{620} &= 2(36agci + 696bfdh + 2340cge^2 + 2876d^2f^2 + 40b^2h^2 \\
&\quad + 544c^2g^2 + 1025e^4 - 388bgch - 340bhe^2 - 2596cgdf \\
&\quad - 4180dfe^2 - 16ahbi + a^2i^2 - 52aidf + 30aie^2 - 60agdh \\
&\quad - 60bfci + 60aeg^2 + 60iec^2 - 360befg - 360cdeh - 40agf^2 \\
&\quad - 40d^2ci + 240bf^3 + 240hd^3 - 420cef^2 - 420egd^2 \\
&\quad + 20ach^2 + 20gib^2 + 20ahef + 20debi + 180bdg^2 + 180fhc^2 \\
&\quad - 100bgef - 100dech), \\
D_{810} &= -2(18aibj + 536bhci + 4256cgdh + 13328defg + 4704e^2f^2 \\
&\quad - a^2j^2 - 49b^2i^2 - 400c^2h^2 - 784d^2g^2 - 40ajch + 56ajdg \\
&\quad - 28ajef - 392bidg + 196bief - 560chef - 32aci^2 \\
&\quad - 32b^2hj + 112aidh + 112cgbj - 224aieg - 224dfbj \\
&\quad + 140aif^2 + 140e^2bj - 896bdh^2 - 896c^2gi + 1792bheg \\
&\quad + 1792dfci - 1120bhf^2 - 1120e^2ci - 6272ceg^2 - 6272d^2fh \\
&\quad + 3920cgf^2 + 3920e^2dh - 7840df^3 - 7840ge^3),
\end{aligned}
$$

* $D_{330} = \frac{3}{2} D_{222} + \frac{1}{2} B^2_6$. † $D_{530} = -\frac{3}{4} D_{620} + \frac{1}{4} B^2_8$.

Equations which determine D_{330} and D_{530}, the quantities by means of which the remaining derivatives of the sixth and eighth orders have been expressed.

$$* D_{621} = -4\,(7bhci + 22cgdh + 39defg + 30e^2f^2 - 2b^2i^2 + 25c^2h^2 - 47d^2g^2 - 2ajch + 7ajdg - 5ajef + 74bgdi - 73befi - 127chef + 2aci^2 + 2b^2hj - 7aidh - 7cgbj - 22aieg - 22dfbj + 25aif^2 + 25bje^2 - 52bdh^2 - 52c^2gi + 23bgeh + 23cfdi + 70bhf^2 + 70ce^2i + 32ceg^2 + 32d^2fh + 25cgf^2 + 25dhe^2 - 50df^3 - 50ge^3 - 45agfh - 45cedj - 45bfg^2 - 45eid^2 + 27aeh^2 + 27c^2fj - 20ag^3 - 20jd^3).$$

We may now proceed to demonstrate an important property of the derivatives of the fourth degree, analogous to the one which exists for the third degree. Let U, V, W, X be functions of any order f: then, investigating the value of the expression

$$B_{2f-2\alpha}\,[B_\alpha(U, V),\ B_\alpha(W, X)].$$

This reduces itself in the first place to

$$\overline{\theta\phi}^{2f-2\alpha}\ \overline{12}^{\alpha}\ \overline{34}^{\alpha}\ UVWX,$$

where ξ_θ, η_θ refer to U and V, and ξ_ϕ, η_ϕ to W and X: this comes to writing $\xi_\theta = \xi_1 + \xi_2$, $\eta_\theta = \eta_1 + \eta_2$, and $\xi_\phi = \xi_3 + \xi_4$, $\eta_\phi = \eta_3 + \eta_4$; whence

$$\overline{\theta\phi} = \overline{13} + \overline{14} + \overline{23} + \overline{24},$$

or the function in question is

$$(13 + 14 + 23 + 24)^{2f-2\alpha}\ \overline{12}^{\alpha}\ \overline{34}^{\alpha}\ UVWX.$$

But all the terms of this where the sum of the indices of ξ_1, η_1 or ξ_2, η_2 or ξ_3, η_3 or ξ_4, η_4, exceed f, vanish: whence it is only necessary to consider those of the form

$$K_r\,(\overline{13}.\overline{42})^r\,(\overline{14}.\overline{23})^{f-\alpha-r}\,(\overline{12}.\overline{34})^{\alpha}\ UVWX,$$

where K_r denotes the numerical coefficient

$$\frac{(-)^r.\,[2f-2\alpha]^{2f-2\alpha}}{[r]^r\,[r]^r\,[f-\alpha-r]^{f-\alpha-r}\,[f-\alpha-r]^{f-\alpha-r}},$$

or $\quad B_{2f-2\alpha}\,[B_\alpha(U, V),\ B_\alpha(W, X)]$

$$= \Sigma\,\{K_r\,D_{\alpha,r,f-\alpha-r}(U, V, W, X)\}.$$

In particular, if $U = V = W = X$, writing also B_α for $B_\alpha(U, U)$,

$$B_{2f-2\alpha}(B_\alpha, B_\alpha) = \Sigma\,(K_r\ D_{\alpha,r,f-\alpha-r}).$$

$$* D_{540} = 2D_{621} + D_{810}.$$

Equation to determinate D_{540}.

If a is odd, this becomes

$$0 = \Sigma (K_r \, D_{a, r, f-a-r}),$$

an equation which must be satisfied identically by the relations that exist between the quantities D. If, on the contrary, a is even, we see that there are as many independent functions of the form

$$B_{2f-2a}(B_a, B_a)$$

as there are of the form D; and that these two systems may be linearly expressed, either by means of the other. Thus, for the orders 3, 5, 7, the derivatives D are respectively equal, neglecting a numerical factor, to

$$B_6(U^2, U^2), \quad B_{10}(U^2, U^2), \quad B_{14}(U^2, U^2).$$

For the sixth order they may be linearly expressed by means of

$$B_{12}(U^2, U^2), \quad B_6^2,$$

and so on. All that remains to complete the theory of the fourth degree is to find the general solution of this system of equations, as also of the system connecting the derivatives D.

Passing on to a more general property. Let $U_1, U_2 \ldots U_p$ be functions of the orders $f_1, f_2 \ldots f_p$; and suppose

$$\Theta(U_2 \ldots U_p) = \square\, U_2 \ldots U_p$$

a function of the degree f_1 in the variables: suppose that $\Theta(U_2 \ldots U_p)$ contains the differential coefficients of the order r_2 for U_2, r_3 for U_3, &c., so that $f_1 = (f_2 - r_2) + \ldots (f_p - r_p)$. Consider the expression

$$B_{f_1}\{U_1, \Theta(U_2 \ldots U_p)\},$$

which reduces itself in the first place to

$$(\overline{12} + \overline{13} \ldots + \overline{1p})^{f_1} \, \square \, U_1 U_2 \ldots U_p,$$

then to $\quad K(\overline{12}^{f_2-r_2} \, \overline{13}^{f_3-r_3} \, \vdots \, \overline{1p}^{f_p-r_p} \, \square \, U_1 U_2 \ldots U_p;$

where for shortness

$$K = \frac{[f_1]^{f_1}}{[f_2 - r_2]^{f_2-r_2} \ldots [f_p - r_p]^{f_p-r_p}}.$$

For if one of the indices were smaller another would be greater, for instance that of $\overline{12}$: and the symbols ξ_2, η_2 in $\overline{12}^{f_2-r_2-\lambda} \, \square$ would rise to an order higher than f_2, or the term would vanish. Hence, writing

$$\square' = \overline{12}^{f_2-r_2} \, \overline{13}^{f_3-r_3} \ldots \overline{1p}^{f_p-r_p}$$

and $\quad \Theta'(U_1, U_2, \ldots, U_p) = \square' \, U_1 U_2 \ldots U_p),$

we have $B_{f_1}\{U_1, \Theta(U_2 \ldots U_p)\} = K\Theta'(U_1, U_2 \ldots U_p)$;
i.e. the first side is a constant derivative of $U_1, U_2 \ldots U_p$.

Suppose $$U_1 = \frac{1}{[f_1]^{f_1}}(a_0 x^{f_1} + \ldots),$$

$$\Theta(U_2 \ldots U_p) = \frac{1}{[f_1]^{f_1}}(A_0 x^{f_1} + \ldots),$$

then $$K\Theta'(U_1 \ldots U_p) = a_0 A_{f_1} - \frac{f_1}{1} a_1 A_{f_1-1} + \ldots;$$

i.e. $A_{f_1} = K\dfrac{d}{da_0}\Theta'(U_1 \ldots U_p)$, $\dfrac{f_1}{1}A_{f_1-1} = K\dfrac{d}{da_1}\Theta'(U_1, U_2 \ldots U_p) \ldots$
or finally,

$$\Theta(U_2 \ldots U_p) = \frac{K}{[f_1]^{f_1}}\left(x^{f_1}\frac{d}{da_{f_1}} - x^{f_1-1}y\frac{d}{da_{f_1-1}} + \ldots\right)\Theta'(U_1 \ldots U_p),$$

an equation which holds good; changing, however, the numerical factor, when several of the functions $U_1 \ldots U_p$ become identical. Hence the theorem: if U be a function given by

$$U = \frac{1}{[f]^f}(a_0 x^f + a_1 x^{f-1}y + \ldots),$$

and Θ be any constant derivative whatever of U, then

$$\left(x^f\frac{d}{da_f} - x^{f-1}y\frac{d}{da_{f-1}} + \ldots\right)\Theta$$

is a derivative of U, and its value, neglecting a numerical factor, may be found by omitting in the symbol $\square$, which corresponds to the derivative Θ, the factors which contain any one, no matter which, of the symbolic numbers. (See Note.*)

If, for example,

$$-\tfrac{1}{2}D_{210} = \Theta = 6abcd - 4ac^3 - 4bd^3 + 3b^2c^2 - a^2d^2,$$

or $$\square = \overline{12}^2 . \overline{34}^2 . \overline{13} . 42;$$

then $$\left(x^3\frac{d}{da} - x^2y\frac{d}{dc} + xy^2\frac{d}{db} - y^3\frac{d}{da}\right)\Theta$$

reduces itself, omitting a numerical factor, to

$$\overline{12}^2\,\overline{13}\,UUU = -\tfrac{1}{2}B_1\{U, B_1(U,U)\}.$$

This may be compared with some formulæ of M. Eisenstein's, (*Crelle*, XXVII.); adopting his notation, we have

---

* Not given with the present paper.

$$\Phi = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$
$$F = \tfrac{1}{36} B_2(\Phi, \Phi) = (ac - b^2)\, x^2 + (ad - bc)\, xy + (bd - c^2)\, y^2,$$
$$\Phi = -\tfrac{1}{2}\left(x^3 \frac{d}{da} - x^2y \frac{d}{dc} + xy^2 \frac{d}{db} - y^3 \frac{d}{da}\right) D,$$

where D is the same as Θ. Hence to the system of formulæ which he has given, we may add the two following:

$$\Phi_1 = \tfrac{1}{3}\left(\frac{d\Phi}{dx}\frac{dF}{dy} - \frac{d\Phi}{dy}\frac{dF}{dx}\right),$$
$$\Phi_1 = -\tfrac{1}{216}\left\{\frac{d^3\Phi}{dx^3}\frac{d^2\Phi}{dx^2}\frac{d\Phi}{dy} - \frac{d^3\Phi}{dx^2\,dy}\left(2\frac{d^2\Phi}{dx\,dy}\frac{d\Phi}{dy} + \frac{d^2\Phi}{dy^2}\frac{d\Phi}{dx}\right)\right.$$
$$\left. + \frac{d^3\Phi}{dx\,dy^2}\left(2\frac{d^2\Phi}{dx\,dy}\frac{d\Phi}{dx} + \frac{d^2\Phi}{dx^2}\frac{d\Phi}{dy}\right) - \frac{d^3\Phi}{dy^3}\frac{d^2\Phi}{dx^2}\frac{d\Phi}{dx}\right\},$$

the first of which explains most simply the origin of the function Φ_1.

It will be sufficient to indicate the reductions which may be applied to derivatives of the form

$$C_{\alpha,\beta,\gamma}(U, V, W) = \overline{23}^{\alpha} . \overline{31}^{\beta} . \overline{12}^{\gamma}\; UVW,$$

where U, V, W are *homogeneous* functions. In fact, if

$$\xi x + \eta y = \Xi,$$

the above becomes, neglecting a numerical factor,

$$(\Xi_1 . \overline{23})^{\alpha}\, (\Xi_2 . \overline{31})^{\beta}\, (\Xi_3 . \overline{12})^{\gamma}\; UVW,$$

where the symbols ξ, η are supposed not to affect the x, y which enter into the expressions Ξ. But we have identically

$$\Xi_1\overline{23} + \Xi_2\overline{31} + \Xi_3\overline{12} = 0,$$

an equation which gives rise to reductions similar to those which have been found for the derivatives $D_{\alpha,\beta,\gamma}$, but which require to be performed with care, in order to avoid inaccuracies with respect to the numerical factors. It may, however, be at once inferred, that the number of independent derivatives $D_{\alpha,\beta,\gamma}$ is the same with that of the independent derivatives $D_{\alpha,\beta,\gamma}$ for the same value of α, β, γ.

From similar reasonings to those by which $B\{U, B(U,U)\}$ has been found, the following general theorem may be inferred.

"The derivative of any number of the derivatives of one or more functions, or even of any number of functions of these derivatives, is itself a derivative of the original functions."

For the complete reduction of these double derivatives, it would be sufficient, theoretically, to be able to reduce to the smallest number possible, the derivatives of any given degree whatever. This has been done for the derivatives of the third degree $C_{\alpha,\beta,\gamma}$, and for those of the fourth degree, in which all the differentiations rise to the same order $(D_{\alpha,\beta,\gamma})$: it seems, however, very difficult to extend these methods even to the next simplest cases,—extensive researches in the theory of the division of numbers would probably be necessary. Important results might be obtained by connecting the theory of hyperdeterminants with that of elimination, but I have not yet arrived at anything satisfactory upon this subject. I shall conclude with the remark, that it is very easy to find a series, or rather a series of series's of hyperdeterminants of all degrees, viz. the determinants

$$\begin{vmatrix} a, & b \\ b, & c \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c \\ b, & c, & d \\ c, & d, & e \end{vmatrix}, \quad \begin{vmatrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \\ d, & e, & f, & g \end{vmatrix} \ \&c.$$

$$\begin{vmatrix} . & a, & b, & c \\ . & b, & c, & d \\ a, & b, & c & . \\ b, & c, & d & . \end{vmatrix}, \quad \begin{vmatrix} . & a, & b, & c, & d, & e \\ . & b, & c, & d, & e, & f \\ . & c, & d, & e, & f, & g \\ a, & b, & c, & d, & e & . \\ b, & c, & d, & e, & f & . \\ c, & d, & e, & f, & g & . \end{vmatrix} \ \&c. \quad \begin{vmatrix} . & . & a, & b, & c, & d \\ . & . & b, & c, & d, & e \\ . & a, & b, & c, & d & . \\ . & b, & c, & d, & e & . \\ a, & b, & c, & d & . & . \\ b, & c, & d, & e & . & . \end{vmatrix} \ \&c.$$

However, these functions are not all independent; *e.g.* the last may be linearly expressed by the square of the second and the cube of $(ae - 4bd + 3c^2)$: nor do I know the symbolical form of these hyperdeterminant determinants.

INVESTIGATION OF PROPERTIES OF THE HYPERBOLA.

By EDMOND R. TURNER, Caius College.

THE following is a mode of treating the hyperbola, in a manner similar to that given by Mr. O'Brien in a former number of the Journal. It does not require the use of an imaginary angle or quantity, as the method does which is given in his treatise on Analytical Geometry.

It is evident that we may put the equation to the hyperbola in the form of two equations, by means of a subsidiary angle ϕ, by assuming

$$x = a \sec \phi, \quad y = b \tan \phi;$$

and if ϕ and ϕ' be the angles corresponding to P and D, the extremities of two conjugate diameters (ϕ' being a subsidiary angle, for expressing the equation to the conjugate hyperbola)

$$\phi = \phi'.$$

Since the equation to the conjugate hyperbola is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,$$

we must for this curve assume

$$y = b \sec \phi', \quad x = a \tan \phi'.$$

Let xy, $x'y'$ be the points P and D respectively, then we have

$$\left.\begin{aligned} x &= a \sec \phi, \quad & y &= b \tan \phi \\ x' &= a \tan \phi', \quad & y' &= b \sec \phi' \end{aligned}\right\} \dots\dots\dots (1);$$

but,

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 0;$$

therefore, substituting from (1),

$$\sin \phi - \sin \phi' = 0,$$

or

$$\phi = \phi'.$$

If $r\theta$, $r'\theta'$ be the polar coordinates of P and D,

$$r^2 = x^2 + y^2 = a^2 \sec^2 \phi + b^2 \tan^2 \phi,$$

$$r'^2 = x'^2 + y'^2 = a^2 \tan^2 \phi + b^2 \sec^2 \phi;$$

therefore

$$r^2 - r'^2 = a^2 - b^2.$$

Also if A be the area of the parallelogram completed upon CP and CD,

$$\begin{aligned} A &= rr' \sin (\theta' - \theta) \\ &= xy' - x'y \\ &= ab \sec^2 \phi - ab \tan^2 \phi \\ &= ab. \end{aligned}$$

If in the expressions $x = a \sec \phi$, $y = b \tan \phi$, we put ϕ' for ϕ, and change x and y into x' and y', therefore

$$x' = a \tan \phi' = a \tan \phi = \frac{a}{b}\, y,$$

$$y' = b \sec \phi' = b \sec \phi = \frac{b}{a}\, x.$$

To find the evolute to the hyperbola.
The equation to the normal, being

$$y - y_1 = -\frac{dx_1}{dy_1}(x - x_1),$$

may be written

$$y - b\tan\phi = -\frac{a}{b}\sin\phi\,(x - a\sec\phi),$$

or
$$y = -x\frac{a}{b}\sin\phi + \frac{a^2+b^2}{b}\tan\phi;$$

therefore, differentiating with regard to ϕ,

$$0 = -x\frac{a}{b}\cos\phi + \frac{a^2+b^2}{b}\sec^2\phi:$$

therefore
$$\sec^3\phi = \frac{x}{\alpha}; \quad \text{where } \alpha = \frac{a^2+b^2}{a}.$$

Changing $\sec\phi, x, y, a, b$ into $\tan\phi, y, x, b, a$, respectively,

$$\tan^3\phi = \frac{y}{\beta}, \quad \text{where } \beta = \frac{a^2+b^2}{b};$$

therefore
$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1.$$

It is evident that, if a circle be described on the axis major, and a tangent be drawn to it from the foot of the ordinate at any point; the radius passing through the point of contact will be inclined at the angle ϕ to the axis of abscissas.

NOTE ON THE RINGS AND BRUSHES IN THE SPECTRA PRODUCED BY BIAXAL CRYSTALS.

By William Thomson, B.A.

It has been shown in this Journal (vol. III. p. 286) that if any system of isothermal plane curves be given, the orthogonal system, which is proved to be necessarily isothermal also, may in every case be determined. Thus if $v = a$ be the equation to the first system, v being a function of x and y which satisfies the equation

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0,$$

we shall have for the equation to the orthogonal system

$$u = \int\left(\frac{dv}{dy}\,dx - \frac{dv}{dx}\,dy\right) = \beta,$$

the expression under the sign of integration being in this case a complete differential, and the equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$$

will be satisfied; results which may be readily verified.

To take an example, let Q, Q', &c. be any number of fixed points determined by the coordinates (a, b), (a', b'), &c., and let r, r', &c. be the distances of the point $P(xy)$ from those points. We may take for the equation of an isothermal system of curves

$$v = m \log r + m' \log r' + \&c. = \alpha \;\ldots\ldots\; (1),$$

where $r^2 = (x - a)^2 + (y - b)^2$, &c., and m, m', &c. are constants.

In this case we have

$$u = m \tan^{-1}\frac{x - a}{y - b} + m' \tan^{-1}\frac{x - a'}{y - b'} + \&c. = \beta \;\ldots\; (2)$$

for the orthogonal system, which, as may be readily verified, is also isothermal.

To take a simple case, let there be only two fixed points, Q, Q', and let $m = m' = 1$. The equation of the first system becomes

$$rr' = \epsilon^{\alpha} = c \;\ldots\ldots\ldots\ldots\ldots\; (3);$$

and, if we take the origin as the point of bisection of QQ', and make this line the axis of x, the equation of the second system becomes

$$\tan^{-1}\frac{x - a}{y} + \tan^{-1}\frac{x + a}{y} = \beta \;\ldots\ldots\ldots\; (4),$$

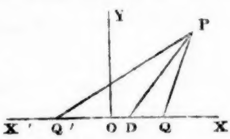
or

$$\frac{2xy}{y^2 - x^2 + a^2} = \tan\beta,$$

which may be put under the form

$$x^2 + 2kxy - y^2 = a^2 \;\ldots\ldots\ldots\ldots\; (5).$$

The equation (3) represents the series of *lemniscates* which Herschel has shown to be the forms of the rings in a biaxal crystal. Also (4) is the equation of a brush, since, if we draw PD bisecting the angle QPQ' and meeting QQ' in D, we have



$$PDQ = PQX - DPQ = PQ'X + Q'PD$$

$$= \tfrac{1}{2}(PQX + PQ'X) = \tfrac{1}{2}\left(\tan^{-1}\frac{x - a}{y} + \tan^{-1}\frac{x + a}{y}\right).$$

Hence (4) represents the locus of the points P, when the angle PDQ is constant, which is the characteristic property of a brush.

Thus we see that the rings in a biaxal crystal form a system of isothermal plane curves, and the brushes the conjugate orthogonal system.

Some curious properties of the second system, which is a series of hyperbolas, may be deduced from equation (5). Let $\frac{a}{h_1^{\frac{1}{2}}}$ and $\frac{a}{h_2^{\frac{1}{2}}}$ be the semiaxes, real and imaginary, and θ the angle which the former makes with OX. To determine h_1, h_2, and θ, we have

$$(h-1)(h+1) = k^2,$$

$$\tan^2\theta = \frac{h_1 - 1}{h_1 + 1},$$

from which we deduce $h_1 = (k^2+1)^{\frac{1}{2}}$,

$$h_2 = -(k^2+1)^{\frac{1}{2}},$$

$$\tan^2\theta = \frac{(k^2+1)^{\frac{1}{2}} - 1}{(k^2+1)^{\frac{1}{2}} + 1}.$$

Hence $$(k^2+1)^{\frac{1}{2}} = \frac{1+\tan^2\theta}{1-\tan^2\theta} = \frac{1}{\cos 2\theta},$$

and $$\left(\frac{a}{h_1^{\frac{1}{2}}}\right)^2 = a^2\cos 2\theta.$$

Thus the second system is a series of rectangular hyperbolas whose vertices lie on the lemniscate of which the equation is

$$\rho^2 = a^2 \cos 2\theta.$$

By putting $y = 0$ in (5), we have, for the two values of x, $\pm a$, and therefore each hyperbola passes through the points Q and Q'. The series is determined by this and the preceding property.

In addition it may be remarked that, by putting $\epsilon^2 = a^2$ in (3), we find $$(x^2+y^2)^2 = 2a^2(x^2-y^2),$$

or, in polar coordinates,

$$\rho^2 = 2a^2\cos 2\theta,$$

for the equation of one of the curves of the first system. This curve is a lemniscate similar to that which is the locus of the vertices of the second system, and similarly situated, but of different magnitude.

ON THE PRINCIPAL AXES OF A RIGID BODY.

By William Thomson.

§ 1. *Definition of principal axes of a solid body.*—2...5. *Centrifugal forces generated by revolution round a fixed axis.* 6, 7. *Determination of the principal axes through any point.* 8. *Demonstration that they are perpendicular to one another.* 9. *Determination of principal moments.*—10, 11. *Examination of cases in which there are principal axes in any of the coordinate planes.*—12. *Conditions that the axes of coordinates may be principal axes.*

In the following paper no new results are brought forward, but the method of investigation is in some respects simpler than that which is usually given in treatises on Dynamics, and I am not aware that the definition which I have adopted has been employed by other writers.

1. The definition of principal axes usually given in treatises on Dynamics involves the consideration of three at right angles to one another, and it is not easy from it to deduce a simple definition of a single principal axis through any point. The dynamical property which is proved, that the principal axes through a point are the only "permanent axes of rotation" of the body when held by the point, affords a very simple definition of a principal axis, which may be stated as follows:—

Def. *A principal axis through any point of a solid body is a line such that, if the body revolve round it, the centrifugal forces generated are either in equilibrium with one another, or have a single resultant passing through the point.*

2. Let OM be any axis through a point O of a rigid body, determined relatively to three rectangular axes OX, OY, OZ, by the direction cosines l, m, n, and let the body revolve round OM with an angular velocity ω. Let P be any point (x, y, z) and let PM be drawn perpendicular to OM. The centrifugal force on an element $\delta\mu$ of the body at P will be $\omega^2 PM\delta\mu$, and its component in any direction will be found by substituting for PM in this expression its corresponding projection. Now the coordinates of P, relatively to M, are

$$x - l(lx + my + nz),\ y - m(lx + my + nz),\ z - n(lx + my + nz).$$

Hence the components of the centrifugal force on $\delta\mu$, parallel to OX, OY, OZ, are

$$\left.\begin{aligned}&\omega^2\delta\mu\,\{x - l(lx + my + nz)\}\\&\omega^2\delta\mu\,\{y - m(lx + my + nz)\}\\&\omega^2\delta\mu\,\{z - n(lx + my + nz)\}\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ (1).$$

The components of the couple obtained by transferring the force on $\delta\mu$ to the origin are consequently

$$\left.\begin{array}{l}\omega^2\delta\mu\,(mz - ny)(lx + my + nz)\\ \omega^2\delta\mu\,(nx - lz)(lx + my + nz)\\ \omega^2\delta\mu\,(ly - mx)(lx + my + nz)\end{array}\right\}\ldots\ldots\ldots\ldots (2).$$

To get the components (X, Y, Z) of the resultant force at the origin, and the components (L, M, N) of the resultant couple of the centrifugal forces, we must take the sum of the preceding expressions for every element of the body. Thus, if μ be the mass of the body and $\bar{x}, \bar{y}, \bar{z}$ the coordinates of the centre of gravity; and if

$$U = \Sigma\delta\mu\, x\,(lx + my + nz),$$
$$V = \Sigma\delta\mu\, y\,(lx + my + nz),$$
$$W = \Sigma\delta\mu\, z\,(lx + my + nz),$$

we have

$$\left.\begin{array}{l}X = \omega^2\mu\,\{\bar{x} - l\,(l\bar{x} + m\bar{y} + n\bar{z})\}\\ Y = \omega^2\mu\,\{\bar{y} - m\,(l\bar{x} + m\bar{y} + n\bar{z})\}\\ Z = \omega^2\mu\,\{\bar{z} - n\,(l\bar{x} + m\bar{y} + n\bar{z})\}\end{array}\right\}\ldots\ldots (3),$$

$$\left.\begin{array}{l}L = \omega^2\,(mW - nV)\\ M = \omega^2\,(nU - lW)\\ N = \omega^2\,(lV - mU)\end{array}\right\}\ldots\ldots\ldots\ldots (4).$$

If we denote the sums

	$\Sigma\delta\mu x^2$,	$\Sigma\delta\mu y^2$,	$\Sigma\delta\mu z^2$,	$\Sigma\delta\mu yz$,	$\Sigma\delta\mu zx$,	$\Sigma\delta\mu xy$
by	F	G	H	A'	B'	C'

we shall have the following expressions for U, V, W,

$$\left.\begin{array}{l}U = Fl + C'm + B'n\\ V = C'l + Gm + A'n\\ W = B'l + A'm + Hn\end{array}\right\}\ldots\ldots\ldots\ldots (5).$$

If we denote the moments of inertia of the body round the axes of coordinates by A, B, C, so that

$$A = \Sigma\delta\mu\,(y^2 + z^2) = G + H,\quad B = H + F,\quad C = F + G,$$

we have for L, M, N, the modified expressions

$$\left.\begin{array}{ll} & L = \omega^2\,(nv - mw)\\ & M = \omega^2\,(lw - nu)\\ & N = \omega^2\,(mu - lv)\\ \text{where} & u = Al - C'm - B'n\\ & v = -C'l + Bm - A'n\\ & w = -B'l - A'm + Cn\end{array}\right\}\ldots\ldots\ldots\ldots (6).$$

3. Comparing the values of X, Y, Z, given by equations (3), with the expressions (1), we conclude that the absolute amount of the centrifugal force is the same as if the whole mass of the body were collected at its centre of gravity; but equations (4) shew that its moments are in general not the same as they would be in this case, and that there is not generally a single resultant of the centrifugal forces.

4. From equations (3) and (4) we deduce the relations

$$\left.\begin{aligned} lX + mY + nZ &= 0 \\ lL + mM + nN &= 0 \end{aligned}\right\} \dots\dots\dots\dots (7),$$

and hence the resultant force and the axis of the resultant couple are each perpendicular to the axis of rotation. These conclusions might have been anticipated from the circumstance that each component of the centrifugal force is in a line perpendicular to the axis of rotation, and passing through it.

5. If O and $\overline{M}$ be fixed points in the axis, and $O\overline{M}$ be given, the equations (3) and (4) enable us to find the pressure which the body, when revolving uniformly, exerts upon them. If we take $l = 0$, $m = 0$, $n = 1$, so that $\overline{M}$ may be a point in OZ, we obtain the ordinary expressions for the pressure on a fixed axis.

If the axes of coordinates be such that

$$A' = 0, \quad B' = 0, \quad C' = 0,$$

(principal axes according to the ordinary definition), the expressions for the components of the centrifugal couple become much simplified. Thus if we take the equations (6) we have, in this case,

$$\begin{aligned} L &= \omega^2 (B - C)\, mn, \\ M &= \omega^2 (C - A)\, nl, \\ N &= \omega^2 (A - B)\, lm, \end{aligned}$$

which are the expressions indicated by Poinsot in his memoir on Rotatory Motion.*

6. If the centrifugal forces are either in equilibrium or have a single resultant passing through the origin, the couples L, M, N must vanish. Hence the conditions that OM may be a principal axis, according to the definition stated above, are

* For analytical demonstrations of the various theorems indicated by Poinsot in this Memoir, reference may be made to an elegant paper in Liouville's Journal, entitled "*Thèse sur le mouvement d' un corps solide autour d' un point fixe, par M. Briot, Professeur au Collège Royale d' Orléans.*"

$$\left.\begin{aligned} m(B'l + A'm + Hn) - n(C'l + Gm + A'n) &= 0 \\ n(Fl + C'm + B'n) - l(B'l + A'm + Hn) &= 0 \\ l(C'l + Gm + A'n) - m(Fl + C'm + B'n) &= 0 \end{aligned}\right\} \ldots\ldots (8),$$

or

$$\left.\begin{aligned} n(-C'l + Bm - A'n) - m(-B'l - A'm + Cn) &= 0 \\ l(-B'l - A'm + Cn) - n(Al - C'm - B'n) &= 0 \\ m(Al - C'm - B'n) - l(-C'l + Bm - A'n) &= 0 \end{aligned}\right\} \ldots (9);$$

the first set being obtained from equations (4), and the second, which differ only in form, from (6). In the investigation of principal axes commonly given, the quantities F, G, H are made use of instead of the moments of inertia, A, B, C; but as by using the latter quantities a geometrical representation of the analysis, by means of Poinsot's "momental ellipsoid", may be introduced, (a different "ellipsoid of construction" being made use of in the ordinary method,) in the present article equations (9) will be employed.

If none of the quantities l, m, n vanish, (the cases in which the equations are satisfied when one or more of these quantities vanish will be examined below,) the three equations (9), on account of the peculiar form of their first members, which satisfy the relation $Ll + Mm + Nn = 0$, will be equivalent to two, which may be written thus:

$$\frac{Al - C'm - B'n}{l} = \frac{-C'l + Bm - A'n}{m} = \frac{-B'l - A'm + Cn}{n} \ldots (10).$$

Hence we infer that the diameters of the surface

$$Ax^2 + By^2 + Cz^2 - 2A'yz - 2B'zx - 2C'xy = D \ldots. (11),$$

which cut their diametral planes at right angles, are principal axes of the solid.*

7. The two equations (10) are sufficient for determining the ratios $l : m : n$; but it is convenient to assume a third unknown quantity P, to represent each member of the equations. We thus obtain the three equations

$$\left.\begin{aligned} Pl &= Al - C'm - B'n \\ Pm &= -C'l + Bm - A'n \\ Pn &= -B'l - A'm + Cn \end{aligned}\right\} \ldots\ldots\ldots (12).$$

Eliminating $l : m : n$ in the usual manner, we find

$$(A-P)(B-P)(C-P) \\ - A'^2(A-P) - B'^2(B-P) - C'^2(C-P) - 2A'B'C' = 0 \ldots (13),$$

* The same property may be proved of the surface

$$Fx^2 + Gy^2 + Hz^2 + 2A'yz + 2B'zx + 2C'xy = D,$$

by using equations (8) instead of (9).

a cubic equation to determine P, which may be shewn to have three real roots. Substituting for P in (12) any one of these values, we have three linear equations, from any two of which the same values of the ratios $l:m:n$ may be obtained. Hence there are three principal axes through O.

8. We may shew that these are at right angles to one another in the following manner.*

Let l_1, m_1, n_1, and l_2, m_2, n_2, be the direction-cosines of any two principal axes, and P_1, P_2 the corresponding roots of the cubic. Then, substituting in (12) the values l_1, m_1, n_1, P_1, for l, m, n, P, multiplying the first equation by l_2, the second by m_2, and the third by n_2, and adding, we find

$$P_1(l_1l_2 + m_1m_2 + n_1n_2) = Q,$$

where $Q = Al_1l_2 + Bm_1m_2 + Cn_1n_2$
$$- A'(m_1n_2 + n_1m_2) - B'(n_1l_2 + l_1n_2) - C'(l_1m_2 + m_1l_2).$$

Commencing with the system l_2, m_2, n_2, P_2, and following a similar process, we should have found

$$P_2(l_1l_2 + m_1m_2 + n_1n_2) = Q,$$

on account of the symmetry of Q. Hence, by subtraction,

$$(P_1 - P_2)(l_1l_2 + m_1m_2 + n_1n_2) = 0;$$

and therefore, unless P_1 be equal to P_2,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0;$$

which shews that any two principal axes, corresponding to different roots of the discriminating cubic, are at right angles.

9. The moment of inertia of the body round any axis may, as is well known, be expressed in terms of the sums or integrals A, B, C, A', B', C', and the quantities which determine the position of the axis, and therefore may be found without farther summation or integration. Thus, if P be the moment of inertia round an axis (l, m, n) through O, we have

$$\begin{aligned} P &= \Sigma\delta\mu\,\{(mz-ny)^2 + (nx-lz)^2 + (lx-my)^2\} \\ &= l^2\,\Sigma\delta\mu\,(y^2+z^2) + m^2\,\Sigma\delta\mu\,(z^2+x^2) + n^2\,\Sigma\delta\mu\,(x^2+y^2) \\ &\qquad - 2mn\,\Sigma\delta\mu\,yz - 2nl\,\Sigma\delta\mu\,zx - 2lm\,\Sigma\delta\mu\,xy \end{aligned}$$

or $$P = Al^2 + Bm^2 + Cn^2 - 2A'mn - 2B'nl - 2C'lm \ldots (14).$$

* This demonstration was first given by Cauchy.

Hence, the moment of inertia of the body about any axis OQ, cutting the surface (11) in Q, is $\frac{D}{OQ^2}$; whence the surface is called *the momental ellipsoid.*

Each member of equations (10) is found, by multiplying numerators and denominators by l, m, n, and adding respectively, to be equal to

$$Al^2 + Bm^2 + Cn^2 - 2A'mn - 2B'nl - 2C'lm.$$

Hence, if (l, m, n) be a principal axis, the moment of inertia, given by (14), is the quantity P, of which the values are determined by the discriminating cubic, and therefore the three principal moments of inertia, relative to the point O, are the roots of the cubic equation (13).

10. Let us now consider the case in which equations (9) can be satisfied when one of the quantities l, m, n vanishes. Thus, if $n = 0$, the equations become

$$m(lB' + mA') = 0,$$
$$l(lB' + mA') = 0,$$
$$m(lA - mC') - l(-lC' + mB) = 0 \quad \ldots\ldots (a).$$

Since l and m cannot both vanish when $n = 0$, the first two equations give

$$lB' + mA' = 0 \quad \ldots\ldots\ldots\ldots\ldots (b).$$

Eliminating $l : m$ from (a) by means of this, we obtain

$$B'(AA' + B'C') - A'(BB' + C'A') = 0 \quad \ldots\ldots (c),$$

which is therefore the condition that there may be a principal axis in the plane (xy). Its position in the plane will be given by equation (b), unless both A' and B' vanish; in which case, the quadratic (a) giving two values of $\frac{l}{m}$, and (b) being identically true, there would be two principal axes in the plane (xy).

11. Let us next suppose both m and n to vanish, and investigate under what conditions equations (9) can be satisfied. In this case the first will be identically true, and the second and third will give

$$B' = 0, \quad C' = 0 \quad \ldots\ldots\ldots\ldots (a);$$

which are therefore the conditions that OX may be a principal axis. The remaining principal axes, which will lie in the plane (yz), will be found by making B', C', and l

vanish in equations (9). The second and third will thus be satisfied identically, and the first will become

$$n(Bm - A'n) - m(-A'm + Cn) = 0;$$

which will determine two axes at right angles, being in fact the equation for determining the principal axes of the ellipse in which the surface (11) cuts the plane of yz.

12. From equations (a) (§ 11) it follows, that if OX, OY, OZ be principal axes, we must have $A' = 0$, $B' = 0$, $C' = 0$, or

$$\Sigma\, \delta\mu\, yz = 0, \quad \Sigma\, \delta\mu\, zx = 0, \quad \Sigma\, \delta\mu\, xy = 0.$$

These equations are usually given as the definition of a system of principal axes.

In the next Number another method of treating the equations of condition for principal axes, corresponding to the method followed in a paper "*On the Reduction of the General Equation of the Second Degree*," (1st Series, vol. IV. p. 227); and application will be made to the investigation of the relations which exist between the principal axes through different points of a solid.

Glasgow, Jan. 6, 1846.

ON CIRCULAR SECTIONS OF THE LOCUS OF THE GENERAL EQUATION OF THE SECOND ORDER.

By the Rev. PERCIVAL FROST, M.A.

LET the equation to the surface be

$$u = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy$$
$$+ 2a''x + 2b''y + 2c''z + d = 0 \ldots (a).$$

Let l, m, n be the direction-cosines of a plane cutting the surface in a circle, f, g, h the coordinates of its centre, and l', m' n' the direction cosines of any diameter of this circle, of which the equations will consequently be

$$\frac{x-f}{l'} = \frac{y-g}{m'} = \frac{z-h}{n'} = r \ldots\ldots\ldots (b).$$

Therefore at the points of intersection with the surface we have, by combining these with equation (a),

$$Mr^2 + 2Nr + P = 0,$$

the roots of which must be equal and of opposite signs: hence $N = 0$; and, since P is independent of l', m', n', M

must have the same value for all values of l', m', n' consistent with the equations

$$\left.\begin{aligned} ll' + mm' + nn' &= 0 \\ \text{and} \qquad l'^2 + m'^2 + n'^2 &= 1 \end{aligned}\right\} \ldots\ldots\ldots\ldots (1).$$

If u' be the value of u when f, g, h are written for x, y, z, we have

$$N = l' \frac{du'}{df} + m' \frac{du'}{dg} + n' \frac{du'}{dh} =$$

and therefore, since l', m', n' are indeterminate,

$$\frac{du'}{ldf} = \frac{du'}{mdg} = \frac{du'}{ndh} \ldots\ldots\ldots\ldots (2).$$

Also $\quad M = al'^2 + bm'^2 + cn'^2 + 2a'm'n' + 2b'n'l' + 2c'l'm'.$

Let $\quad a - M = \alpha, \quad b - M = \beta, \quad c - M = \gamma.$

Then, since $M = M(l'^2 + m'^2 + n'^2)$, the preceding equation may be put under the form

$$\alpha l'^2 + \beta m'^2 + \gamma n'^2 + 2a'm'n' + 2b'n'l' + 2c'l'm' = 0 \ldots (c).$$

Multiplying by n^2 and substituting from the first of equations (1), we have

$$(\alpha l'^2 + \beta m'^2)\, n^2 + \gamma\,(ll' + mm')^2 + 2c'l'm'n^2$$
$$- 2n\,(a'm' + b'l')\,(ll' + mm') = 0 \ldots (d).$$

Since (c) must be satisfied for all values of the ratio $l' : m' : n'$ consistent with the first of equations (1), in the latter equation (d) we may equate the coefficients of l'^2, $l'm'$, and n'^2 to zero. Hence

$$\alpha n^2 - 2b'ln + \gamma l^2 = 0 \ldots\ldots\ldots\ldots (3),$$
$$\gamma m^2 - 2a'mn + \beta n^2 = 0 \ldots\ldots\ldots\ldots (4),$$

and

$$a'ln + b'mn - \gamma lm - c'n^2 = 0 \ldots\ldots\ldots\ldots (5).$$

Substituting in (5) for a' and b' their values by (3) and (4), we obtain

$$\beta l^2 - 2c'lm + \alpha m^2 = 0 \ldots\ldots\ldots\ldots (6).$$

To eliminate the ratios $l : m : n$ from the equations (3), (4), and (6), we have, by (3) and (4),

$$(\alpha a'^2 + \beta b'^2)\, n^2 + \gamma\,(a'^2 l^2 + b'^2 m^2) = 2a'b'n\,(a'l + b'm)$$
$$= 2a'b'\,(\gamma' m + c'n^2) \text{ by } (5).$$

Hence $\quad (\alpha a'^2 + \beta b'^2 - 2a'b'c')\, n^2 + \gamma\,(a'l - b'm)^2 = 0 \ldots (e).$

By (3) and (4),

$$(\alpha m^2 - \beta l^2)\, n^2 + 2lmn\,(a'l - b'm) = 0,$$

and by (6), $\quad (\alpha m^2 + \beta l^2)^2 = 4c'^2 l^2 m^2,$

which gives $\quad (\alpha m^2 - \beta l^2)^2 = 4\,(c'^2 - \alpha\beta)\, l^2 m^2.$

Therefore $\quad (a'l - b'm)^2 = (c'^2 - \alpha\beta)\, n^2.$

Hence, by comparison with (e),

$$(\alpha a'^2 + \beta b'^2 + \gamma c'^2 - 2a'b'c' - \alpha\beta\gamma)\, n^2 = 0,$$

and therefore

$$\alpha a'^2 + \beta b'^2 + \gamma c'^2 - 2a'b'c' - \alpha\beta\gamma = 0 \;\ldots\ldots (7).$$

Substituting for α, β, γ their values, we have

$$(M-a)(M-b)(M-c) - a'^2(M-a) - b'^2(M-b) - c'^2(M-c) \\ - 2a'b'c' = 0 \ldots (8),$$

a cubic equation to determine M, which has necessarily three real roots. Substituting any one of them for M in (3), (4), and (6), we obtain three equations, any two of which would give the same values of $l:m:n$, thus fixing the positions of the planes of circular section.* Equations (2) will give the loci of their centres. If two of the equations be used, we should obtain four systems of values of $l:m:n$; of these the two only which are compatible with the third equation must be taken.

Cor. 1. If $a' = b' = c' = 0$, equation (8) becomes

$$(M-a)(M-b)(M-c) = 0.$$

If we take $M = b$, (4) and (5) will become

$$\gamma m^2 = 0, \quad \text{and} \quad \alpha m^2 = 0:$$

therefore $$m = 0,$$

* [The elimination of l, m, n between the equations (3), (4), (6) has been given by Mr. Sylvester (first series, vol. II. p. 233), as an example of the dialytic method according to which, in this case, he forms equation (5) and the two which correspond in symmetry, and eliminates $l^2, m^2, n^2, mn, nl, lm$ from the six equations, in which they enter linearly.

Equations (7) may also be found by expressing the condition that the cone

$$\alpha x^2 + \beta y^2 + \gamma z^2 + 2a'yz + 2b'zx + 2c'xz = 0 \;\ldots\ldots\ldots\ldots (f),$$

may have a plane for one sheet. Since the surface is of the second order, it follows that it must consist of two planes, and therefore one root of its discriminating cubic must vanish, which is expressed by (7). Either of these planes would be parallel to (real or imaginary) circular sections of the given surface, and consequently for each root of the cubic in M there are two, andonly two (both real or both imaginary), systems of circular sections. The intersection of the pair of planes corresponding to one value of M will be the principal axis of the surface (f) which corresponds to the root zero of the discriminating cubic. Hence, by the formulæ in a paper "On the Reduction of the General equation of the Second Degree," (first series vol. IV. p. 227), if l, m, n be the direction-cosines of either plane, we have

$$\frac{l}{\alpha a' - b'c'} + \frac{m}{\beta b' - c'a'} + \frac{n}{\gamma c' - a'b'} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (g),$$

and it may be verified that this equation, and one of the equations (3), (4), (6), would, by elimination, lead to the two others, if the relation (7) be attended to. Equation (g) along with any one of the three equations in the text gives the two required systems of values of $l:m:n$, without ambiguity.]

and
$$\frac{l^2}{a-b} = \frac{n^2}{b-c},$$

and the locus of the centres is given by equations

$$bg + b'' = 0,$$

and
$$\frac{af + a''}{(a-b)^{\frac{1}{2}}} = \frac{ch + c''}{(b-c)^{\frac{1}{2}}}.$$

The values of l and n are impossible unless b be intermediate between a and c, and therefore the planes which cut the surface in circles are perpendicular to the plane of greatest and least axes.

Cor. 2. If $a' = c' = 0$,

the cubic becomes

$$(M - b)\{(M - a)(M - c) - b'^2\} = 0.$$

I. Let $M = b$,

$$\gamma m^2 = 0, \quad m = 0,$$
$$(b - c) l^2 + 2b'ln + (b - a) n^2 = 0;$$

therefore $(b - c)(b - a) < b'^2$ if the values of $\dfrac{l}{n}$ be real, and different.

II.
$$(M - a)(M - c) = b'^2$$
$$\frac{l^2}{a} = -\frac{m^2}{\beta} = \frac{n^2}{\gamma},$$

hence $\dfrac{\beta}{a}$ must be negative, and therefore the quadratic in M must have a root between b and a, which requires that

$$(b - a)(b - c) > b'^2.$$

When this is satisfied there will be one root between b and that one of the quantities a, c which differs least from b, and the others will lie between a and c. The former root will make both $\dfrac{\beta}{a}$ and $\dfrac{\beta}{\gamma}$ negative, and the latter will make one positive and the other negative. Hence the principal sections corresponding to the former are real, and to the latter, imaginary.

Cor. 3. In the case of a surface of revolution the roots of equations (3), (4), (6) are equal; therefore

$$a\gamma = b'^2,$$
$$\beta\gamma = a'^2$$
$$a\beta = c'^2.$$

Hence
$$\gamma = \frac{a'b'}{c'},$$

and
$$M = c - \frac{a'b'}{c'} = b - \frac{c'a'}{b'} = a - \frac{b'c'}{a'} \;\ldots\ldots\; (a).$$

The axis of revolution will be the locus of the centres and its equations will be found as follows:

Since
$$\frac{b'c'}{a'} m^2 - 2c'lm + \frac{a'c'}{b'} l^2 = 0,$$

we have
$$a'l = b'm = c'n;$$

and therefore
$$a'(af + c'g + b'h + a'') = b'(c'f + bg + a'h + b'')$$
$$= c'(b'f + a'g + ch + c''),$$

which are the required equations. These may be modified as follows:
$$(aa' - b'c')\left(f + \frac{a'a''}{aa' - b'c'}\right) = (bb' - a'c')\left(g + \frac{b'b''}{bb' - a'c'}\right).$$

Hence, by (a),
$$a'\left(f + \frac{a'a''}{aa' - b'c'}\right) = b'\left(g + \frac{b'b''}{bb' - c'a'}\right) = c'\left(h + \frac{c'c''}{cc' - a'b'}\right),$$

which are the required equations in their simplest forms.

Cambridge, Jan. 28, 1846.

ON SYMBOLICAL GEOMETRY.

By Professor Sir William Rowan Hamilton, LL.D.

(*Continued from* p. 57.)

Determinateness of the first Four Operations on Geometrical Fractions (or Quotients).

8. Meanwhile the principles and definitions which have been already laid down, are sufficient to conduct to clear and determinate interpretations of all operations of combining geometrical quotients among themselves, by any number of additions, subtractions, multiplications, and divisions: each *quotient* of the kind here mentioned being regarded, by what has been already shown, as the *mark of a certain complex relation between two straight lines in space,* depending not only on their *relative lengths,* but also on their *relative*

directions. If we denote now by a symbol of fractional form, such as $\frac{\mathrm{b}}{\mathrm{a}}$, the quotient thus obtained by dividing one line b by another line a, when directions as well as lengths are attended to, the definitional equations (26), (27), (28), (29), will take these somewhat shorter forms:*

$$\frac{\mathrm{c}}{\mathrm{a}} + \frac{\mathrm{b}}{\mathrm{a}} = \frac{\mathrm{c} + \mathrm{b}}{\mathrm{a}}; \qquad \frac{\mathrm{c}}{\mathrm{a}} - \frac{\mathrm{b}}{\mathrm{a}} = \frac{\mathrm{c} - \mathrm{b}}{\mathrm{a}}; \ldots . \ (46),$$

$$\frac{\mathrm{c}}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{c}}{\mathrm{b}}; \qquad \frac{\mathrm{c}}{\mathrm{a}} \div \frac{\mathrm{b}}{\mathrm{a}} = \frac{\mathrm{c}}{\mathrm{b}}; \ \ldots\ldots \ (47),$$

which agree in all respects with the corresponding formulæ of ordinary algebra, and serve to fix, in the present system, the meanings of the operations $+$, $-$, $\times$, $\div$, on what may be called *geometrical fractions*. These FRACTIONS being only other forms for what we have called *geometrical quotients* in earlier articles of this paper, we may now write the identity,

$$\frac{\mathrm{b}}{\mathrm{a}} = \mathrm{b} \div \mathrm{a} \ \ldots\ldots\ldots\ldots\ldots \ (48).$$

* On the principles alluded to in former notes, the formulæ for the addition, subtraction, multiplication, and division, of any two geometrical fractions, might be thus written:

$$\frac{D - C}{B - A} + \frac{C - A}{B - A} = \frac{D - A}{B - A},$$

$$\frac{D - A}{B - A} - \frac{C - A}{B - A} = \frac{D - C}{B - A},$$

$$\frac{D - A}{C - A} \times \frac{C - A}{B - A} = \frac{D - A}{B - A},$$

$$\frac{D - A}{B - A} \div \frac{C - A}{B - A} = \frac{D - A}{C - A};$$

A, B, C, D being symbols of any four points of space, and B − A being a symbol of the straight line drawn to B from A. If we denote this line by the biliteral symbol BA, we obtain the following somewhat shorter forms, which do not however all agree so closely with the forms of ordinary algebra:

$$\frac{DC}{BA} + \frac{CA}{BA} = \frac{DA}{BA},$$

$$\frac{DA}{BA} - \frac{CA}{BA} = \frac{DC}{BA},$$

$$\frac{DA}{CB} \times \frac{CA}{BA} = \frac{DA}{BA},$$

$$\frac{DA}{BA} \div \frac{CA}{BA} = \frac{DA}{CA}.$$

For the same reason, an *equation between any two such fractions*, for example the following,

$$\frac{f}{e} = \frac{b}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (49),$$

is to be understood as signifying, 1st, that the *length* of the one *numerator* line f is to the length of its own *denominator* line e *in the same ratio* as the length of the other numerator line b to the length of the other denominator line a; 2nd, that these four lines are *co-planar*, that is to say, in or parallel to one common plane; and 3rd, that the *same amount and direction of rotation*, round an axis perpendicular to this common plane, which would bring the line a into the direction originally occupied by b, would also bring the line e into the original direction of f. The same complex relation between the same four lines may also (by what has been already seen) be expressed by the *inverse* equation

$$\frac{e}{f} = \frac{a}{b} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (50),$$

or by the *alternate* form $\quad \frac{f}{b} = \frac{e}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (51).$

Two fractions which are, in this sense, *equal* to the same third fraction, are also equal to each other; and the *value* of such a fraction is not altered by altering the lengths of its numerator and denominator in any common ratio; nor by causing both to turn together through any common amount of rotation, in a common direction, round an axis perpendicular to both; nor by transporting either or both, without rotation, to any other positions in space. When the lengths and directions of any three co-planar lines, a, b, e, are given, it is always possible to determine the length and direction of a fourth line f, which shall be co-planar with them, and shall satisfy an equation between fractions, of the form (49). It is therefore possible to *reduce any two geometrical fractions to a common denominator;* or to satisfy not only the equation (49), but also this other equation,

$$\frac{h}{g} = \frac{c}{a} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (52),$$

by a suitable choice of the three lines a, b, c, when the four lines e, f, g, h, are given; since, whatever may be the given directions of these four lines, it is always possible to find (or to conceive as found) a fifth line a, which shall be at once co-planar with the pair e, f, and also with the pair g, h.

For a similar reason it is always possible to transform two given geometrical fractions into two others equivalent to them, in such a manner, that the new denominator of one shall be equal to the new numerator of the other; or to satisfy the two equations

$$\frac{h}{g} = \frac{c'}{a'}, \quad \frac{f}{e} = \frac{a'}{b'} \ldots\ldots\ldots\ldots (53),$$

by a suitable choice of the three lines a′, b′, c′, whatever the four given lines e, f, g, h, may be. Making then for abridgment

$$c + b = d, \quad c - b = d' \ldots\ldots\ldots\ldots (54),$$

and interpreting a sum or difference of lines as has been done in former articles, we see that it is always possible to choose eight lines a, b, c, d, a′, b′, c′, d′, so as to satisfy the conditions (49), (52), (53), (54); and thus, by (46) and (47), to interpret the sum, the difference, the product, and the quotient of *any two* given geometrical fractions, $\frac{f}{e}$ and $\frac{h}{g}$, as being each equal to *another given fraction* of the same sort, as follows:

$$\frac{h}{g} + \frac{f}{e} = \frac{d}{a}, \quad \frac{h}{g} - \frac{f}{e} = \frac{d'}{a} \ldots\ldots\ldots (55),$$

$$\frac{h}{g} \times \frac{f}{e} = \frac{c'}{b'}, \quad \frac{h}{g} \div \frac{f}{e} = \frac{c}{b} \ldots\ldots\ldots (56),$$

any variations in the new numerators and denominators, which are consistent with the foregoing conditions, being easily seen to make no changes in the values of the fractions which result. The *interpretations* of those four symbolic combinations, which are the first members of the four equations (55) and (56), are thus entirely *fixed:* and we are *no longer at liberty, in the present system,* to introduce arbitrarily any *new meanings* for those symbolic forms, or to subject them to any *new laws* of combination among themselves, without examining whether such meanings or such laws are consistent with the principles and definitions which it has been thought right to establish already, as appearing to be more simple and primitive, and more intimately connected with the application of symbolical language to geometry, or at least with the plan on which it is here attempted to make that application, than any of those other laws or meanings. If, for example, it shall be found that, in virtue of the foregoing principles, the *successive addition* of any number of geometrical fractions gives a result which is independent of

their order, this consequence will be, for us, a *theorem*, and not a definition. And if, on the contrary, the same principles shall lead us to regard the *multiplication* of geometrical fractions as being in general a *non-commutative* operation, or as giving a result which is *not* independent of the order of the factors, we shall be obliged to accept this conclusion also, that we may preserve consistency of system.

Separation of the Scalar and Vector parts of Sums and Differences of Geometrical Fractions.

9. To develope the geometrical meaning of the first equation (46), we may conceive each of the two numerator lines b, c, and also their sum d, to be orthogonally projected, first on the common denominator line a itself, and secondly on a plane perpendicular to that denominator. The former projections may be called b_1, c_1, d_1; the latter, b_2, c_2, d_2; and thus we shall have the nine relations,

$$\left.\begin{array}{lll} b_2 + b_1 = b, & b_1 \parallel a, & b_2 \perp a, \\ c_2 + c_1 = c, & c_1 \parallel a, & c_2 \perp a, \\ d_2 + d_1 = d, & d_1 \parallel a, & d_2 \perp a, \end{array}\right\} \dots\dots (57),$$

together with the three equations

$$c + b = d, \quad c_1 + b_1 = d_1, \quad c_2 + b_2 = d_2 \dots (58);$$

of which the two last are deducible from the first, by the geometrical properties of projections. We have, therefore, by (46),

$$\frac{c}{a} + \frac{b}{a} = \frac{d}{a} = \frac{d_2}{a} + \frac{d_1}{a} \dots\dots\dots\dots (59),$$

$$\frac{d_1}{a} = \frac{c_1}{a} + \frac{b_1}{a}, \quad \frac{d_2}{a} = \frac{c_2}{a} + \frac{b_2}{a} \dots\dots\dots (60).$$

Since the three projections b_1, c_1, d_1, are parallel to a (in that sense of the word *parallel* which does not exclude coincidence), the three quotients in the first equation (60) are what we have already named *scalars;* that is, they are what are commonly called real numbers, positive, negative, or zero: they are also the scalar parts of the three quotients in the first equation (59), so that we may write

$$\frac{b_1}{a} = S\frac{b}{a}, \quad \frac{c_1}{a} = S\frac{c}{a}, \quad \frac{d_1}{a} = S\frac{d}{a} \dots\dots (61),$$

using the letter S here, as in a former article, for the characteristic of the operation of *taking the scalar part* of any geometrical quotient, or fraction. (If any confusion should

be apprehended, on other occasions, from this use of the letter S, and if the abridged word Scal. should be thought too long, the sign $\underset{c}{\mathrm{S}}$ might be employed.) Eliminating the four symbols b_1, c_1, d_1, d, between the first equation (59), the first equation (60), and the three equations (61), we obtain the result

$$\mathrm{S}\left(\frac{c}{a}+\frac{b}{a}\right)=\mathrm{S}\frac{c}{a}+\mathrm{S}\frac{b}{a}\,\ldots\ldots\ldots\ldots(62);$$

in which, by the foregoing article, $\frac{b}{a}$ and $\frac{c}{a}$ may represent any two geometrical fractions: so that we may write generally

$$\mathrm{S}\left(\frac{h}{g}+\frac{f}{e}\right)=\mathrm{S}\frac{h}{g}+\mathrm{S}\frac{f}{e}\,\ldots\ldots\ldots\ldots(63),$$

and may enunciate in words the same result by saying, that the *scalar of the sum* of any two such fractions is equal to the *sum of the scalars*. In like manner, the three other projections b_2, c_2, d_2, being each perpendicular to a, the three other partial quotients, which enter into the second equation (60), are what we have already called *vectors* in this paper, or more fully they are the vector parts of the three quotients in the first equation (59); so that we may write

$$\frac{b_2}{a}=\mathrm{V}\frac{b}{a},\quad \frac{c_2}{a}=\mathrm{V}\frac{c}{a},\quad \frac{d_2}{a}=\mathrm{V}\frac{d}{a}\,\ldots\ldots(64),$$

V being here used, as in a former article, for the characteristic of the operation of *taking the vector part;* we have, therefore,

$$\mathrm{V}\left(\frac{c}{a}+\frac{b}{a}\right)=\mathrm{V}\frac{c}{a}+\mathrm{V}\frac{b}{a}\,\ldots\ldots\ldots(65),$$

$$\mathrm{V}\left(\frac{h}{g}+\frac{f}{e}\right)=\mathrm{V}\frac{h}{g}+\mathrm{V}\frac{f}{e}\,\ldots\ldots\ldots(66),$$

and may assert that the *vector of the sum* of any two geometrical fractions is equal to the *sum of the vectors.* These formulæ (63) and (66) are important in the present system; they are however, as we see, only symbolical expressions of those very simple geometrical principles from which they have been derived, through the medium of the equations (58); namely, the principles that, *whether on a line or on a plane*, the *projection of a sum* of lines is equal to the *sum of the projections*, if the word *sum* be suitably interpreted. The analogous interpretation of a *difference* of lines, combined with similar considerations, gives in like manner the formulæ

$$\mathrm{S}\left(\frac{\mathrm{h}}{\mathrm{g}}-\frac{\mathrm{f}}{\mathrm{e}}\right)=\mathrm{S}\,\frac{\mathrm{h}}{\mathrm{g}}-\mathrm{S}\,\frac{\mathrm{f}}{\mathrm{e}}\,\ldots\ldots\ldots\ldots\,(67),$$

$$\mathrm{V}\left(\frac{\mathrm{h}}{\mathrm{g}}-\frac{\mathrm{f}}{\mathrm{e}}\right)=\mathrm{V}\,\frac{\mathrm{h}}{\mathrm{g}}-\mathrm{V}\,\frac{\mathrm{f}}{\mathrm{e}}\,\ldots\ldots\ldots\,(68);$$

that is to say, the *scalar and vector of the difference* of any two geometrical fractions are respectively equal to the *differences of the scalars and of the vectors* of those fractions; precisely as, and because, the *projection of a difference* of two lines, whether on a line or on a plane, is equal to the *difference of the projections*.

Addition and Subtraction of Vectors by their Indices.

10. We see, then, that in order to combine by addition or subtraction any two geometrical fractions, it is sufficient to combine separately their scalar and their vector parts. The former parts, namely the scalars, are simply *numbers*, of the kind called commonly real; and are to be added or subtracted among themselves according to the usual rules of algebra. But for effecting with convenience the combination of the latter parts among themselves, namely the vectors, which have been shown in a former article to be of a kind essentially distinct from all stages of the progression of real number from negative to positive infinity (and therefore to be rather *extra-positives* than either positive or *contra-positive* numbers), it is necessary to establish other rules: and it will be found useful for this purpose to employ the consideration of certain connected *lines*, namely the *indices*, of which each is determined by, and in its turn completely characterises, that vector quotient or fraction to which it corresponds, according to the construction assigned in the 7th article. If we apply the rules of that construction to determine the indices of the vector parts of any two fractions and of their sum, we may first, as in recent articles, reduce the two fractions to a common denominator; and may, for simplicity, take this denominator line a of a length equal to that assumed unit of length which is to be employed in the determination of the indices. Then, having projected, as in the last article, the new numerators b and c, and their sum d, on a plane perpendicular to a, and having called these projections b_2, c_2, d_2, as before; we may conceive a right-handed rotation of each of these three projected lines, through a right angle, round the line a as a common axis, which shall transport them without altering their lengths or relative di-

rections, and therefore without affecting their mutual relation as summands and sum, into coincidence with three other lines b_3, c_3, d_3, such that

$$d_3 = c_3 + b_3 \ldots\ldots\ldots\ldots\ldots (69);$$

and these three new lines will be the three indices required. For a right-handed rotation through a right angle, round the line b_3 as an axis, would bring the line a into the direction originally occupied by b_2; and the length of b_2 is to the length of a in the same ratio as the length of b_3 to the assumed unit of length; therefore b_3 is, in the sense of the 7th article, the index of the vector quotient $\frac{b_2}{a}$, that is, the index of the vector part of the fraction $\frac{b}{a}$, or $\frac{f}{e}$; and similarly for the indices of the two other fractions, in the first equation (59). We may therefore write, as consequences of the construction lately assigned, and of the equations (49) and (52),

$$b_3 = I\frac{f}{e}; \quad c_3 = I\frac{h}{g}; \quad d_3 = I\left(\frac{h}{g} + \frac{f}{e}\right) \ldots\ldots (70);$$

if we agree for the present to prefix the letter I to the symbol of a geometrical fraction, as the characteristic of the operation of *taking the index of the vector part.* Eliminating now the three symbols b_3, c_3, d_3 between the four equations (69) and (70), we obtain this general formula:

$$I\left(\frac{h}{g} + \frac{f}{e}\right) = I\frac{h}{g} + I\frac{f}{e} \ldots\ldots\ldots\ldots (71),$$

which may be thus enunciated: the *index of the vector part of the sum* of any two geometrical fractions is equal to the *sum of the indices* of the vector parts of the summands. Combining this result with the formula (63), which expresses that the scalar of the sum is the sum of the scalars, we see that the complex *operation of adding any two geometrical fractions*, of which each is determined by its scalar and by the index of its vector part, may be in general *decomposed into two* very simple but *essentially distinct operations;* namely, *first*, the operation of adding together *two numbers*, positive or negative or null, so as to obtain a third number for their sum, according to the usual rules of elementary algebra; and *second*, the operation of adding together *two lines* in space, so as to obtain a third line, according to the geometrical rules of the composition of motions, or by drawing the diagonal of a parallelogram. In like manner the operation of *taking the difference* of two

fractions may be decomposed into the two operations of taking separately the difference of two numbers, and the difference of two lines; for we can easily prove that

$$\mathrm{I}\left(\frac{h}{g}-\frac{f}{e}\right)=\mathrm{I}\,\frac{h}{g}-\mathrm{I}\,\frac{f}{e}\,\ldots\ldots\ldots\ldots(72);$$

or, in words, that the *index* (of the vector part) *of the difference* of any two fractions is equal to the *difference of the indices.* And because it has been seen that not only for numbers but also for lines, considered among themselves, any number of summands may be in any manner grouped or transposed without altering the sum; and that the sum of a scalar and a vector is equal to the sum of the same vector and the same scalar, combined in a contrary order; it follows that the *addition* of any number of geometrical fractions is an *associative* and also a *commutative* operation: in such a manner that we may now write

$$\frac{h}{g}+\frac{f}{e}=\frac{f}{e}+\frac{h}{g};\ \frac{k}{i}+\left(\frac{h}{g}+\frac{f}{e}\right)=\left(\frac{k}{i}+\frac{h}{g}\right)+\frac{f}{e}=\frac{f}{e}+\frac{h}{g}+\frac{k}{i},\ \&c.\ \ldots\ldots\ldots(73),$$

whatever straight lines in space may be denoted by e, f, g, h, i, k, &c. We may also write, concisely,

$$\mathrm{S}\Sigma=\Sigma\mathrm{S};\quad \mathrm{V}\Sigma=\Sigma\mathrm{V};\quad \mathrm{I}\Sigma=\Sigma\mathrm{I}\ldots\ (74);$$

$$\mathrm{S}\Delta=\Delta\mathrm{S};\quad \mathrm{V}\Delta=\Delta\mathrm{V};\quad \mathrm{I}\Delta=\Delta\mathrm{I}\ldots.(75);$$

using Σ, Δ as the characteristics of sum and difference, while S, V, I are still the signs of scalar, vector, index.

Separation of the Scalar and Vector Parts of the Product of any two Geometrical Fractions.

11. The definitions (46), (47) of addition and multiplication of fractions, namely

$$\frac{c}{a}+\frac{b}{a}=\frac{c+b}{a},\qquad \frac{c}{a}\times\frac{a}{b}=\frac{c}{b},$$

give obviously, for any 4 straight lines a, b, c, a′, the formula

$$\left(\frac{c}{a}+\frac{b}{a}\right)\times\frac{a}{a'}=\frac{c+b}{a'}=\left(\frac{c}{a}\times\frac{a}{a'}\right)+\left(\frac{b}{a}\times\frac{a}{a'}\right)\ldots\ldots\ (76);$$

and this other formula of the same kind,

$$\frac{a'}{a}\times\left(\frac{c}{a}+\frac{b}{a}\right)=\frac{a'}{\dfrac{a}{c+b}\times a}=\left(\frac{a'}{a}\times\frac{c}{a}\right)+\left(\frac{a'}{a}\times\frac{b}{a}\right)\ldots.\ (77),$$

may be proved without difficulty to be a consequence of the same definitions; the operation of multiplying a line, by the quotient of two others with which it is co-planar, being interpreted by the definition (23), so as to give, in the present notation,

$$\frac{\mathrm{e}}{\mathrm{a}} \times \mathrm{a} = \mathrm{e} \ldots\ldots\ldots\ldots\ldots\ldots (78).$$

In fact, if we assume, as we may, seven new lines, $\mathrm{d b' c' d' b'' c'' d''}$, so as to satisfy the seven conditions

$$\left.\begin{array}{l} \mathrm{c} + \mathrm{b} = \mathrm{d},\ \dfrac{\mathrm{b}}{\mathrm{a}} = \dfrac{\mathrm{a}}{\mathrm{b}'},\ \dfrac{\mathrm{c}}{\mathrm{a}} = \dfrac{\mathrm{a}}{\mathrm{c}'},\ \dfrac{\mathrm{d}}{\mathrm{a}} = \dfrac{\mathrm{a}}{\mathrm{d}'}, \\ \dfrac{\mathrm{b}''}{\mathrm{a}'} = \dfrac{\mathrm{a}'}{\mathrm{b}'},\ \dfrac{\mathrm{c}''}{\mathrm{a}'} = \dfrac{\mathrm{a}'}{\mathrm{c}'},\ \dfrac{\mathrm{d}''}{\mathrm{a}'} = \dfrac{\mathrm{a}'}{\mathrm{d}'}, \end{array}\right\} \ldots (79),$$

we shall have the first member of the formula (77) equal to $\frac{\mathrm{a}'}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{d}'} = \frac{\mathrm{a}'}{\mathrm{d}'} =$ the second member of that formula; it will therefore be equal to $\frac{\mathrm{d}''}{\mathrm{a}'}$, and consequently will be shown to be $= \frac{\mathrm{c}''}{\mathrm{a}'} + \frac{\mathrm{b}''}{\mathrm{a}'} = \frac{\mathrm{a}'}{\mathrm{c}'} + \frac{\mathrm{a}'}{\mathrm{b}'} =$ the third member of that formula, if we can show that the conditions (79) give the relation

$$\mathrm{d}'' = \mathrm{c}'' + \mathrm{b}'' \ldots\ldots\ldots\ldots\ldots\ldots (80).$$

Now those conditions show that the line a is common to the planes of b, b′, and c, c′, and that it bisects the angle between b and b′, and also the angle between c and c′; therefore the mutual inclination of the lines b′ and c′ is equal to the mutual inclination of b and c; while the lengths of the two former lines are, by the same conditions, inversely proportional to those of the two latter. And on pursuing this geometrical reasoning, in combination with the definitional meanings of the symbolic equations (79), it appears easily that the mutual inclinations of the lines b″, c″, d″, are equal to those of b′, c′, d′, and therefore to those of b, c, d; while the lengths of b″, c″, d″ are inversely proportional to those of b′, c′, d′, and therefore directly proportional to the lengths of b, c, d: since then the line d is the symbolic sum of b and c, or the diagonal of a parallelogram described with those two lines as adjacent sides, it follows that the line d″ is similarly related to b″ and c″, or that the relation (80) holds good. The formula (77) is therefore shown to be true: and although we have not *yet* proved that the multiplication of two geometrical fractions is *always* a *distributive* operation, we see at least that either

factor may be distributed into two partial factors, and that the sum of the two partial products will give the total product, whenever either total factor and the two parts of the other factor are *co-linear;* that is, whenever the planes of these three fractions are *parallel to any common line*, such as the line a in the formulæ (76) (77): the *plane* of a geometrical fraction being one which contains or is parallel to the numerator and denominator thereof. A *scalar* fraction, being the quotient of two parallel lines, of which either may be transported without altering its direction to any other position in space while both may revolve together, may be regarded as having an entirely *indeterminate plane*, which may thus be rendered parallel to any arbitrary line; we shall therefore always satisfy the condition of *co-linearity*, by distributing either or both of two factors into their scalar and vector parts, and may consequently write,

$$\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{f}}{\mathrm{e}} = \left(\mathrm{V}\,\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{S}\,\frac{\mathrm{h}}{\mathrm{g}} \times \frac{\mathrm{f}}{\mathrm{e}}\right)$$

$$= \left(\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{V}\,\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{S}\,\frac{\mathrm{f}}{\mathrm{e}}\right)$$

$$= \left(\mathrm{V}\,\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{V}\,\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{V}\,\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{S}\,\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{S}\,\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{V}\,\frac{\mathrm{f}}{\mathrm{e}}\right) + \left(\mathrm{S}\,\frac{\mathrm{h}}{\mathrm{g}} \times \mathrm{S}\,\frac{\mathrm{f}}{\mathrm{e}}\right) \quad \ldots\ldots (81);$$

or more concisely,

$$(\beta + b)(\alpha + a) = \beta\alpha + \beta a + b\alpha + ba \quad \ldots\ldots (82),$$

if we denote, as in a former article, vectors by greek and scalars by italic letters, and omit the mark of multiplication between any two successive letters of these two kinds, or between sums of such letters, when those sums are enclosed in parentheses. But the multiplication of scalars is effected, as we have seen, by the ordinary rules of algebra; and to multiply a vector by a scalar, or a scalar by a vector, is easily shown, by the definitions already laid down, to be equivalent to multiplying by the scalar, on the plan of the sixth article, either the index or the numerator of the vector, without altering the denominator of that vector: thus, in the second member of (82), the term ba is a known scalar, and the terms $b\alpha$, βa are known vectors, if the partial factors a, b, α, β be known; in order therefore to apply the equation (82), which in its form agrees with ordinary algebra, to any question of multiplication of any two geometrical fractions, it is sufficient to know how to interpret generally the remaining term

βa, or the product of one vector by another. For this purpose we may always conceive the index $I\beta$ of the vector β to be the sum of two other indices, which shall be respectively parallel and perpendicular to the index $I\alpha$ of the other vector α, as follows:

$$I\beta' \parallel I\alpha, \quad I\beta'' \perp I\alpha, \quad I\beta'' + I\beta' = I\beta \ldots\ldots (83);$$

and then the vector β itself will be, by the last article, the sum of the two new vectors β' and β'', and the planes of these two new vector fractions will be respectively parallel and perpendicular to the plane of the vector fraction α; consequently, the three fractions β', β'', α will be co-linear, and we shall have, by the principle (76),

$$\beta\alpha = (\beta'' + \beta')\,\alpha = \beta''\alpha + \beta'\alpha \ldots\ldots\ldots (84).$$

The problem of the multiplication of *any two* vectors is thus decomposed into the two simpler problems, of multiplying first *two parallel*, and secondly *two rectangular, vectors* together. If then we merely wish to separate the scalar and the vector parts, it is sufficient to observe that if, in the general formula (47), for the multiplication of any two fractions, we suppose the factors to be parallel vectors, then the line a is perpendicular to both b and c, and is also co-planar with them, so that they are necessarily parallel to each other, and the product $\frac{\mathrm{c}}{\mathrm{b}}$ is a scalar; but if, in the same general formula, we suppose the factors to be rectangular vectors, then the three lines a, b, c are themselves mutually rectangular, and the product of the fractions is a vector. Thus, in the formula (84), the partial product $\beta'\alpha$ is a scalar, but the other partial product $\beta''\alpha$ is a vector: and we may write

$$S.\,\beta\alpha = \beta'\alpha; \quad V.\,\beta\alpha = \beta''\alpha \ldots\ldots\ldots\ldots (85).$$

We may therefore, more generally, under the conditions (83), decompose the formula of multiplication (82) into the two following equations:

$$\left.\begin{aligned} S.(\beta + b)(\alpha + a) &= \beta'\alpha + ba; \\ V.(\beta + b)(\alpha + a) &= \beta''\alpha + \beta a + b\alpha \end{aligned}\right\} \ldots\ldots\ldots (86).$$

Or we may write, for abridgment,

$$c = \beta'\alpha + ba; \quad \gamma = \beta''\alpha + \beta a + b\alpha \ldots\ldots (87);$$

and then we shall have this other equation of multiplication,

$$\gamma + c = (\beta + b)(\alpha + a) \ldots\ldots\ldots\ldots (88).$$

And thus the general *separation of the scalar and vector*

parts of the product of any two geometrical fractions may be effected. But it seems proper to examine more closely into the separate meanings of the two partial products of vectors, denoted here by the two terms $\beta'\alpha$ and $\beta''\alpha$; which will be done in the two following articles.

Products of two Parallel Vectors; Geometrical Representations of the Square Roots of Negative Scalars.

12. It was shown, in the last article, that the product of any two parallel vectors, such as α and β', that is, the product of any two vectors of which the planes or the indices are parallel, is equal to a scalar. By pursuing the reasoning of that article, it is easy to show, farther, that this *scalar product of two parallel vectors* is equal to the *product of the numbers* which express the lengths of the two parallel indices; this numerical product being taken with a *negative* or with a *positive* sign, according as these indices are *similar* or *opposite* in direction. In fact, in the general formula $\frac{\mathrm{c}}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{c}}{\mathrm{b}}$, we have now $\mathrm{b} \perp \mathrm{a}$, $\mathrm{c} \parallel \mathrm{b}$; the length of c is to the length of b, in a ratio compounded of the ratio of the length of c to that of a, and of the ratio of the length of a to that of b; and the direction of c is opposite or similar to that of b, according as the two quadrantal rotations in one common plane, from b to a, and from a to c, are performed right-handedly round the same index, or round opposite indices.

We know then perfectly how to interpret the product of any two parallel vectors; and, as a case of such interpretation, if we agree to say that the product of any two equal fractions is the *square* of either, and to write

$$\frac{\mathrm{b}}{\mathrm{a}} \times \frac{\mathrm{b}}{\mathrm{a}} = \left(\frac{\mathrm{b}}{\mathrm{a}}\right)^2 \quad \text{..............(89)},$$

whatever two lines may be denoted by a and b, we see that, in the present system, the *square of a vector is always a negative scalar*, namely the negative of the square of the number which denotes the length of the index of the vector; in such a manner that, for any vector α, we shall have the equation

$$\alpha^2 = -\bar{\alpha}^2 \quad \text{................(90)},$$

if we agree to denote by the symbol $\bar{\alpha}$ that positive or absolute number which expresses the *length of the index* $\mathrm{I}\alpha$. We have then, reciprocally,

$$\bar{\alpha}^2 = -\alpha^2 \quad \text{...............(91)};$$

and may therefore write

$$\bar{a} = \sqrt{(-a^2)} \ldots\ldots\ldots\ldots\ldots (92),$$

$-a^2$ being here a positive number (because a^2 is negative), and $\sqrt{(-a^2)}$ being its positive or absolute *square root,* which is an entirely *determined* (and real) *number,* when the vector a, or even when the length of its index, is determined. But although we might be led to write, in like manner, from (90), the equation

$$a = \sqrt{(-\bar{a}^2)} \ldots\ldots\ldots\ldots\ldots (93),$$

yet the same principles prove that this expression, which may denote generally any *square root of a negative number,* by a suitable choice of the positive number $\bar{a}$, is equal to a *vector* a, of which the index Ia has indeed a *determined length,* but has an entirely *undetermined direction;* the symbol in the second member of the equation (93) may therefore receive (in the present system) infinitely many different geometrical representations, or constructions, though they have all one common character: and it will be a little more consistent with the analogies of ordinary algebra to write the equation under the form

$$a = (-\bar{a}^2)^{\frac{1}{2}} \ldots\ldots\ldots\ldots\ldots (94),$$

using a fractional exponent which suggests a certain degree of indeterminateness, rather than a radical sign which it is often convenient to restrict to one determined value. Thus, for example, the symbol $(-1)^{\frac{1}{2}}$, or the *square root of negative unity,* will, in the present system, denote, or be geometrically constructed by, *any vector of which the index is equal to the unit of length;* that is, any geometrical fraction of which the numerator and the denominator are lines equal to each other in length, but perpendicular to each other in direction. And we see that the geometrical principle, on which this conclusion ultimately depends, is simply this: that *two successive and similar quadrantal rotations, in any arbitrary plane, reverse the direction* of any straight line in that plane. Mr. Warren, confining himself to the consideration of lines in *one fixed plane,* has been led to attribute to his geometrical representations of the square roots of negative numbers, *one fixed direction,* or rather axis, perpendicular to that other axis on which he represents square roots of positive numbers. And other authors, both before and since the publication of Mr. Warren's work,* seem to have been in like manner

* *Treatise on the Geometrical Representation of the Square Roots of Negative Quantities,* by the Rev. John Warren, Cambridge, 1828. See also Dr. Peacock's Treatises on Algebra, and his Report to the British Association, containing references to other works.

disposed to represent positive or negative numbers by lines in some one direction, or in the direction opposite, but symbols of the form $a\sqrt{(-1)}$ by lines perpendicular thereto. Such is at least the impression on the mind of the present writer, produced perhaps by an insufficient acquaintance with the works of those who have already written on this class of subjects. It will however be attempted to show, in a future article of this paper, that the geometrical fractions which have been called *vectors*, in the present and in former articles, may be symbolically equated to their own indices; and that thus *every straight line having direction in space* may properly be looked upon *in the present system* as *a geometrical representation of a square root of a negative number*; while positive and negative numbers are in the same system regarded indeed as belonging to one common *scale* of progression, from $-\infty$ to $+\infty$, but to a scale which is not to be considered as having any one direction rather than any other, in tridimensional space.

Products of two Rectangular Vectors; Non-commutativeness of the Factors, in the general Multiplication of two Geometrical Fractions.

13. The reasoning by which it was shown, in the 11th article, that the *product* $\beta''\alpha$ *of any two rectangular vectors*, α and β'', is *itself a vector*, may be continued so as to show that the number expressing the length of the index of this vector product is the product of the numbers which express the lengths of the indices of the factors; or that, in a notation similar to one employed in the last article,

$$\overline{\beta''\alpha} = \overline{\beta''}\,\overline{\alpha}, \quad \text{when } \mathrm{I}\beta'' \perp \mathrm{I}\alpha \ldots\ldots (95);$$

and therefore that, by the principle (92), for the same case of *rectangular vectors*, we have the formula

$$\sqrt{\{-(\beta''\alpha)^2\}} = \sqrt{(-\beta''^2)}\,\sqrt{(-\alpha^2)} \ldots\ldots\ldots (96).$$

Also in the general formula of multiplication $\frac{\mathrm{c}}{\mathrm{a}} \times \frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{c}}{\mathrm{b}}$, the three lines a, b, c compose here a rectangular system; and therefore the *index of the product* is parallel to the line a, and is consequently *perpendicular to the indices of the two factors*; $\mathrm{I}.\beta''\alpha$ is therefore perpendicular to both $\mathrm{I}\beta''$ and $\mathrm{I}\alpha$; a conclusion which may be extended by (83) and (85) to the multiplication of *any two vectors*, so that we may write generally,

$$\mathrm{I}.\beta\alpha \perp \mathrm{I}\beta; \quad \mathrm{I}.\beta\alpha \perp \mathrm{I}\alpha \ldots\ldots\ldots\ldots (97).$$

Again, we are allowed to suppose, in applying the same general formula of multiplication to the same case of rectangular vectors, that the index $\mathrm{I}\alpha$ of the multiplicand $\frac{\mathrm{a}}{\mathrm{b}}$ is not only parallel to the line c, but similar (and not opposite) in direction to that line; in such a manner that the rotation round c from b to a is positive: and then the rotation round b from a to c is positive, and so is the rotation round a from c to b, and also that round $-$ a from b to c; therefore the index $\mathrm{I}\beta''$ of the multiplier is similar in direction to $+$ b, and the index $\mathrm{I}.\beta''\alpha$ of the product is similar in direction to $-$ a; consequently *the rotation round the index of the product, from the index of the multiplier to that of the multiplicand, is positive.* And although this last result has only been proved here for the case of two rectangular vectors, yet it may easily be shown, by the principles of the 11th article, to extend to the multiplication of two general geometrical fractions. For, in the notation of that article, γ denoting the vector part of the product of any two such fractions, we have, by (87),

$$\mathrm{I}\gamma = \mathrm{I}.\beta''\alpha + a\mathrm{I}\beta + b\mathrm{I}\alpha \ldots\ldots\ldots (98);$$

$\mathrm{I}\gamma$ is therefore the symbolic sum of $\mathrm{I}.\beta''\alpha$ and of two other lines which are respectively parallel to the indices of the vector parts of the two factors, and which consequently have their sum co-planar with those indices, and therefore also co-planar, by (83), with $\mathrm{I}\beta''$ and $\mathrm{I}\alpha$; consequently $\mathrm{I}\gamma$ and $\mathrm{I}.\beta''\alpha$ both lie at the same side of the plane of $\mathrm{I}\alpha$ and $\mathrm{I}\beta''$; and therefore the rotation round $\mathrm{I}\gamma$, like that round $\mathrm{I}.\beta''\alpha$, from $\mathrm{I}\beta''$ to $\mathrm{I}\alpha$, and consequently from $\mathrm{I}\beta$ to $\mathrm{I}\alpha$, is positive. Hence also the rotation round $\mathrm{I}\beta$ from $\mathrm{I}\alpha$ to $\mathrm{I}\gamma$ is positive; that is to say, in the multiplication of two general geometrical fractions, *the rotation round the index of the vector part of the multiplier, from that of the multiplicand to that of the product, is positive;* from which may immediately be deduced a remarkable consequence, already alluded to by anticipation in the 8th article, namely—that the *multiplication of two general geometrical fractions is not a commutative operation,* or that the *order of the factors is not in general indifferent;* since the index of the vector part of the product lies at one or at the other side of the plane of the indices of the vector parts of the two factors, according as those factors are taken in one or in the other order. We have, for example, by the present article, a relation of *opposition* of signs between the products of two *rectangular* vectors, taken in two opposite

orders; which relation may be expressed by the following *equation of perpendicularity*,

$$\alpha\beta'' = -\beta''\alpha, \quad \text{when } \mathrm{I}\beta'' \perp \mathrm{I}\alpha \;\ldots\ldots\; (99).$$

But in the case where the indices of the vector parts α and β of two fractional factors are *parallel* (which includes the case where either of those indices vanishes, the corresponding factor becoming then a scalar), the part β'' of the vector β vanishes, and the latter vector reduces itself by (83) to its other part β'; so that in *this* case, by the results of the last article, the order of the factors is indifferent, and the operation of multiplication is commutative: and thus we may write, as the *equation of parallelism* between two vectors,

$$\alpha\beta' = \beta'\alpha, \quad \text{when } \mathrm{I}\beta' \parallel \mathrm{I}\alpha \;\ldots\ldots\ldots\; (100).$$

It is easy to infer hence, by (84) and (77), that in the more general case of the multiplication of any two vectors α and β, we may write, instead of (85), the following formulæ for the separation of the scalar and vector parts of the product:

$$\left.\begin{aligned} \mathrm{S}.\beta\alpha &= \tfrac{1}{2}(\beta\alpha + \alpha\beta) = \mathrm{S}.\alpha\beta \\ \mathrm{V}.\beta\alpha &= \tfrac{1}{2}(\beta\alpha - \alpha\beta) = -\mathrm{V}.\alpha\beta \end{aligned}\right\} \;\ldots\ldots\; (101),$$

with corresponding formulæ instead of (86), which give

$$(\beta + b)(\alpha + a) - (\alpha + a)(\beta + b) = \beta\alpha - \alpha\beta\,.\,.(102),$$

the second member of this last equation being a vector different from 0, unless it happen that the planes (or the indices) of the vectors α and β are parallel to each other. Finally, we may here observe that in virtue of the principles and definitions already laid down, *the length of the index* ($\mathrm{I}.\beta\alpha$) *of the vector part of the product of any two vectors bears to the unit of length the same ratio which the area of the parallelogram under the indices* ($\mathrm{I}\beta$ and $\mathrm{I}\alpha$) *of the factors bears to the unit of area;* the *direction* of this index of the product being also (as we have seen) *perpendicular to the plane* of the indices of the factors, and therefore to the plane of the parallelogram under them; and being changed to its own *opposite* when the order of the factors is inverted, which *inversion* of their order may be considered as corresponding to a *reversal of the face* of the parallelogram: for all which reasons, there appears to be a propriety in considered this index of the vector part of a product of any two vectors as a symbolical representation of this parallelogram under the indices of the factors, and in writing the symbolical equation

$$\mathrm{I}.\beta\alpha = \square(\mathrm{I}\beta, \mathrm{I}\alpha) \;\ldots\ldots\ldots\ldots\; (103).$$

It will be remembered that the indices $\mathrm{I}(\beta + \alpha)$, $\mathrm{I}(\beta - \alpha)$, of the sum and difference of the same two vectors, are symbolically equal to two different diagonals of the same parallelogram, by former articles of this paper.

(To be continued.)

ERRATA IN THE PRECEDING PORTION OF THIS PAPER.

In second note, page 47, *for* ordinarily *read* ordinally.
In page 50, after equation (11), *for* theory *read* theorem.
In formula (27), page 53, *for* + *read* ÷.
In page 57, *for* co-planal *read* co-planar.

ON THE INTEGRATION OF CERTAIN DIFFERENTIAL EQUATIONS.

By the Rev. BRICE BRONWIN.

THE equations integrated in this paper are linear, and of the second order. The mode of integration is a little different from the methods hitherto employed. But the object chiefly aimed at is to give the most simple and elegant form to the more complex of the two particular integrals. For this will often admit of very different forms.

The formula $\epsilon^{ax} f\left(\frac{d}{dx} + a\right) y = f\left(\frac{d}{dx}\right) \epsilon^{ax} y$ is well known.

Change ϵ^{x} into x, $\frac{d}{dx}$ into $x\frac{d}{dx}$, and put D for $\frac{d}{dx}$; it becomes

$$x^a f(xD + a)\, y = f(xD)\, x^a y \;\ldots\ldots\ldots\; (a).$$

By this formula

$$Dy = x^{-1}(xD)\, y = (xD+1)\, x^{-1} y;$$

$$D^2 y = (xD+1)\, x^{-1} Dy = (xD+1)\, x^{-1}(xD+1)\, x^{-1} y$$
$$= (xD+1)(xD+2)\, x^{-2} y,$$

$D^3 y = (xD+1)(xD+2)(xD+3)\, x^{-3} y$, &c.; and generally

$$D^n y = (xD+1)(xD+2)\ldots\ldots(xD+n)\, x^{-n} y^* \;\ldots\ldots\; (b).$$

Put in this last $D^{-n} y$ for y, and divide by the operating factors; there results

$$x^{-n} D^{-n} y = (xD+n)^{-1}(xD+n-1)^{-1}\ldots\ldots(xD+1)^{-1} y.$$

Multiply this by x^n; and it becomes in virtue of (a)

$$D^{-n} y = (xD)^{-1}(xD-1)^{-1}\ldots\ldots(xD-n+1)^{-1}\, x^n y \ldots\ldots (c).$$

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* [This agrees with a formula proved in Vol. I. of the 1st Series, p. 282; also in Gregory's *Examples*, p. 31.]

Multiply (b) and (c) by x^p, and change y into $x^q y$; they become

$$\left.\begin{array}{l} x^p D^n x^q y = (xD-p+1)(xD-p+2)\ldots(xD-p+n)x^{p+q-n}y \\ x^p D^{-n} x^q y = (xD-p)^{-1}(xD-p-1)^{-1}\ldots(xD-p-n+1)x^{p+q+n}y \end{array}\right\}\ldots(d).$$

These may be convenient. We will now proceed to integrate a few equations.

Let $\dfrac{d^2y}{dx^2} + qx\dfrac{dy}{dx} + qmy = 0$, ($m$ a positive integer). . . .(1).

Multiplying by x^2, this may be written

$$x^2D^2y + qx^2(xD+m)y = 0;$$

by (a) and (b), or by (d), at once this becomes

$$xD(xD-1)y + (xD+m-2)qx^2y = 0.$$

Make $y = (xD+1)(xD+2)\ldots(xD+m-1)x^{-m+1}u = D^{m-1}u$.

Then $x^2y = (xD-1)(xD)\ldots\ldots(xD+m-3)x^{-m+3}u$.

These values of y and x^2y being put in the preceding equation, after dividing by the factors common to both the terms, it becomes

$$(xD+m-1)x^{-m+1}u + qx^{-m+3}u = 0\ \ldots\ldots\ldots\ (2).$$

I shall recur to this equation presently. It is equivalent to $\dfrac{du}{dx} + qxu = 0$, or $u = C\epsilon^{-\frac{1}{2}qx^2}$, which gives a particular integral of (1). To find the other particular integral, make $y = \epsilon^{-\frac{1}{2}qx^2}z$; and (1) will be transformed into

$$\frac{d^2z}{dx^2} - qx\frac{dz}{dx} + qmz = 0\ \ldots\ldots\ldots\ldots\ (3).$$

Treating this in the same manner, we have

$$xD(xD-1)z - (xD-m-2)qx^2z = 0.$$

Make $z = (xD)^{-1}(xD-1)^{-1}\ldots(xD-m)^{-1}x^{m+1}u = D^{-m-1}u$:

then also $x^2z = (xD-2)^{-1}(xD-3)^{-1}\ldots(xD-m-2)^{-1}x^{m+3}u$.

With these values, the common factors being expunged, the preceding becomes

$$(xD-m-1)x^{m+1}u - qx^{m+3}u = 0,$$

or $\dfrac{du}{dx} - qxu = 0$, $u = C_1\epsilon^{\frac{1}{2}qx^2}$; and therefore $z = C_1D^{-m-1}\epsilon^{\frac{1}{2}qx^2}$.

Hence the complete integral of (1) is

$$y = C\left(\frac{d}{dx}\right)^{m-1}\epsilon^{-\frac{1}{2}qx^2} + C_1\epsilon^{-\frac{1}{2}qx^2}\left(\frac{d}{dx}\right)^{-m-1}\epsilon^{\frac{1}{2}qx^2};$$

that of (3) is given by $z = \epsilon^{\frac{1}{2}qx^2}y$.

We now return to (2). By making the factors divided by to operate upon the second member; that would have been

$$(xD+m-1)x^{-m+1}u+qx^{-m+3}u=(xD+m-2)^{-1}(xD+m-3)^{-1}\ldots$$
$$(xD-1)^{-1}0=x^{-m+2}D^{-m}x^{-2}0=x^{-m+2}D^{-m+1}C.$$

This reduction is made by the second of (d) and one integration. Now this would give at once the complete integral of (1). But it would not be so simple and elegant as the above by a great deal, and simplicity of form is often a matter of great importance.

Let $\quad x^2\dfrac{d^2y}{dx^2}-m\dfrac{dy}{dx}-ry=0,\ r=p(p+1)\ \ldots\ldots\ (4).$

Multiply by x, and as before by the formulæ (a), (b), or (d); we find

$$x\{xD(xD-1)-p(p+1)\}y-xDmy=0,$$

or $\qquad x(xD+p)(xD-p-1)y-xDmy=0,$

or $\quad (xD+p-1)(xD-p-2)xy-(xD)my=0\ \ldots\ (5),$

where we may observe that the case of $p=1$ is integrable immediately.

Make $\quad y=(xD+1)(xD+2)\ldots(xD+p-1)x^{-p+1}u=D^{p-1}u,$

and $\qquad xy=(xD)(xD+1)\ldots(xD+p-2)x^{-p+2}u.$

These values put in (5), it will become, dividing by the common factors,

$$(xD-p-2)x^{-p+2}u-mx^{-p+1}u=0,\text{ or } x^2\frac{du}{dx}-(m+2px)u=0,$$

and $\qquad u=Cx^{2p}\varepsilon^{-\frac{m}{x}}.$

Again make

$$y=(xD)^{-1}(xD-1)^{-1}\ldots.(xD-p-1)^{-1}x^{p+2}u=D^{-p-2}u,$$
$$xy=(xD-1)^{-1}\ldots.(xD-p-2)^{-1}x^{p+3}u.$$

With these values (5) becomes

$$(xD+p-1)x^{p+3}u-mx^{p+2}u=0,$$

or $x^2\dfrac{du}{dx}-\{m-(2p+2)x\}u=0$; which gives $u=C_1x^{-2p-2}\varepsilon^{-\frac{m}{x}}.$

The complete integral of (5) therefore is

$$y=C\left(\frac{d}{dx}\right)^{p-1}x^{2p}\varepsilon^{-\frac{m}{x}}+C_1\left(\frac{d}{dx}\right)^{-p-2}x^{-2p-2}\varepsilon^{-\frac{m}{x}}.$$

Here again the integral is much more simple than it would be if found at once by operating upon 0 with the expunged factors.

Let $x^2 \dfrac{d^2y}{dx^2} + (m - x)\, x \dfrac{dy}{dx} + n\,(m - n - 1)\, y = 0$, n integer...(6).

From this we derive

$$\{xD\,(xD - 1) + mxD + n\,(m - n - 1)\}\; y - x\,(xD)\, y = 0:$$

or by further reduction

$$(xD + n)\,(xD + m - n - 1)\, y - (xD - 1)\, xy = 0.$$

Make $y = (xD + n)^{-1}(xD + n - 1)^{-1}...(xD)^{-1} u = x^{-n}D^{-n-1}x^{-1}u$,

$$xy = (xD + n - 1)^{-1}(xD + n - 2)^{-1}\ldots (xD - 1)^{-1} xu.$$

By the substitution of these values, and dividing by the factors common to both the terms, the last equation becomes

$$(xD + m - n - 1)\, u - xu = 0, \quad \text{or } x\frac{du}{dx} + (m - n - 1 - x)\, u = 0,$$

and $u = Cx^{n-m+1}\varepsilon^x$. There are other forms by which this particular integral might be found, but this is the simplest.

Now make $y = x^{-m}\varepsilon^x z$, and with this value of y (6) will be transformed into

$$x^2 \frac{d^2z}{dx^2} + (x - m)\, x\frac{dz}{dx} + (m - n)\,(n + 1)\, z = 0 \;\ldots\ldots\; (7).$$

This gives

$$\{xD\,(xD - m - 1) + (m - n)\,(n + 1)\}\; z + x\,(xD)\, z = 0,$$

or $$(xD + n - m)\,(xD - n - 1)\, z + (xD - 1)\, xz = 0.$$

In the last make

$$z = (xD - n)\,(xD - n + 1)\ldots(xD - 1)\, u = x^{n+1}D^n x^{n-1} u,$$

$$xz = (xD - n - 1)\,(xD - n - 2)\ldots(xD - 2)\, xu.$$

These values, substituted in the preceding, change it into

$$(xD + n - m)\, u + xu = 0, \quad \text{or } x\frac{du}{dx} + (n - m + x)\, u = 0,$$

$$\text{and } u = C_1 x^{m-n}\varepsilon^{-x},\; z = C_1 x^{n+1}\left(\frac{d}{dx}\right)^n x^{m-n-1}\varepsilon^{-x};$$

therefore $y = Cx^{-n}\left(\dfrac{d}{dx}\right)^{-n-1} x^{n-m}\varepsilon^x + C_1 x^{n-m+1}\varepsilon^x\left(\dfrac{d}{dx}\right)^n x^{m-n-1}\varepsilon^{-x}$.

This is the complete integral of (6), and $z = x^m\varepsilon^{-x}y$ will give that of (7). We might of course have found the complete integral at once here, as before indicated; but our object is to exhibit it in the simplest form.

Let $$(1 - x^2)\frac{d^2y}{dx^2} + p\,(p + 1)\, y = 0 \;\ldots\ldots\ldots\; (8).$$

Multiply by x^2, and as before we find

$$xD(xD-1)\,y - x^2\,\{xD(xD-1) - p(p+1)\}\,y = 0,$$

or $$xD(xD-1)\,y - x^2\,(xD+p)(xD-p-1)\,y = 0,$$

and $$xD(xD-1)\,y - (xD+p-2)(xD-p-3)\,x^2y = 0\ldots(9).$$

Make $y = (xD+1)(xD+2)\ldots(xD+p-1)\,x^{-p+1}u = D^{p-1}u$,

$$x^2y = (xD-1)(xD)\ldots(xD+p-3)\,x^{-p+3}u.$$

Substitute these values in the last, and it becomes

$$(xD+p-1)\,x^{-p+1}u - (xD-p-3)\,x^{-p+3}u = 0,$$

or $$(1-x^2)\frac{du}{dx} + 2pxu = 0, \text{ and } u = C\,(1-x^2)^p.$$

Again make

$$y = (xD)^{-1}(xD-1)^{-1}\ldots\ldots(xD-p-1)^{-1}\,x^{p+2}\,u = D^{-p-2}\,u,$$

$$x^2y = (xD-2)^{-1}(xD-3)^{-1}\ldots\ldots(xD-p-3)^{-1}\,x^{p+4}\,u.$$

Put these values in (9), and it will give

$$(xD-p-2)\,x^{p+2}\,u - (xD+p-2)\,x^{p+4}\,u = 0,$$

or $$(1-x^2)\frac{du}{dx} - (2p+2)\,xu = 0, \text{ and } u = C_1\,(1-x^2)^{-p-1}.$$

Consequently $y = C\left(\dfrac{d}{dx}\right)^{p-1}(1-x^2)^p + C_1\left(\dfrac{d}{dx}\right)^{-p-2}(1-x^2)^{-p-1}$, the complete integral of (8). And this also is much more simple than it would be if found at once. We shall give one example more.

Let $x\dfrac{d^2y}{dx^2} + qx\dfrac{dy}{dx} + qmy = 0$, m a positive integer...(10).

This gives, after multiplying by x,

$$xD(xD-1)\,y + qx\,(xD+m)\,y = 0,$$

or $$xD(xD-1)\,y + (xD+m-1)\,qxy = 0.$$

Make $$y = (xD+1)(xD+2)\ldots(xD+m-1)\,x^{-m+1}\,u = D^{m-1}\,u,$$

$$xy = (xD)(xD+1)\ldots(xD+m-2)\,x^{-m+2}\,u.$$

Substitute these expressions of y and xy in the last, and it gives

$$(xD-1)\,x^{-m+1}\,u + qx^{-m+2}\,u = 0, \quad\text{or } x\frac{du}{dx} + (qx-m)\,u = 0,$$

and $$u = Cx^m\,\epsilon^{-qx}.$$

To find the other particular integral make $y = \epsilon^{-qx}z$, and (10) will be transformed into

$$x\frac{d^2z}{dx^2} - qx\frac{dz}{dx} + qmz = 0 \quad\ldots\ldots\ldots\quad (11).$$

From this we deduce, first

$$xD(xD-1)z - x(xD-m)qz = 0,$$

and then $\quad xD(xD-1)z - (xD-m-1)qxz = 0.$

Make $z = (xD)^{-1}(xD-1)^{-1}\dots(xD-m)^{-1}x^{m+1}u = D^{-m-1}u,$

$$xz = (xD-1)^{-1}(xD-2)^{-1}\dots(xD-m-1)^{-1}x^{m+2}u.$$

Put these values in the above, and we find

$$(xD-1)x^{m+1}u - qx^{m+2}u = 0, \text{ or } x\frac{du}{dx} + (m-qx)u = 0,$$

and $\qquad u = C_1 x^{-m}\epsilon^{qx}:$

whence $\quad y = C\left(\frac{d}{dx}\right)^{m-1} x^m\epsilon^{-qx} + C_1\epsilon^{-qx}\left(\frac{d}{dx}\right)^{-m-1} x^{-m}\epsilon^{qx},$

the complete integral of (10). And $z = \epsilon^{qx}y$ will give that of (11).

The method of integration employed in this paper is well adapted to the integration of a certain class of partial differential equations. I shall give an example in concluding this paper. And here let D_x, D_y stand for $\frac{d}{dx}$ and $\frac{d}{dy}$ respectively.

Let $$\frac{d^2z}{dx^2} - \frac{d^2z}{dy^2} - \frac{2}{x}\frac{dz}{dx} = 0 \quad \dots\dots\dots\dots (12).$$

By (*a*) and (*b*) this may be put under the form

$$(xD_x+1)(xD_x+2)x^{-2}z - (yD_y+1)(yD_y+2)y^{-2}z$$
$$-2(xD_x+2)x^{-2}z = 0,$$

or $\quad (xD_x-1)(xD_x+2)x^{-2}z - (yD_y+1)(yD_y+2)y^{-2}z = 0.$

Make $\qquad z = (xD_x-1)u;$

then $\quad x^{-2}z = (xD_x+1)x^{-2}u, \; y^{-2}z = (xD_x-1)y^{-2}u.$

Putting these values in the preceding, and dividing by $xD_x - 1$, it becomes

$$(xD_x+1)(xD_x+2)x^{-2}u - (yD_y+1)(yD_y+2)y^{-2}u$$
$$= (xD_x-1)^{-1}0 = xD^{-1}x^{-2}0 = Cx = xf(y),$$

by (*d*) and by integration. But this is equivalent to

$$\frac{d^2u}{dx^2} - \frac{d^2u}{dy^2} = xf(y).$$

Make $u = v - xD_y^{-2}f(y)$, and we have $\frac{d^2v}{dx^2} - \frac{d^2v}{dy^2} = 0.$

Consequently $\quad v = \phi(y+x) + \psi(y-x),\; z = (xD_x-1)u$

$$= (xD_x-1)v - (xD_x-1)xD_y^{-2}f(y) = (xD_x-1)v = x\frac{dv}{dx} - v;$$

or $z = x\{\phi'(y+x) - \psi'(y-x)\} - \{\phi(y+x) + \psi(y-x)\}.$

If we had not operated upon the cypher, we should have obtained the same result; but we should not have had the same assurance of the generality of the solution.

I presume this method may be applied to similar equations in finite differences.

Gunthwaite Hall, Penistone, March, 1846.

ON THE THEORY OF MAGIC SQUARES, CUBES, &c.

By R. Moon, M.A., Fellow of Queens' College.

In a former paper, published in this *Journal*, I endeavoured to develope a new method of treating the subject of Magic Squares, and exhibited more or less fully the mode of its application to the case of squares containing an odd number of places. On the present occasion I purpose to shew that the same method may be applied to the composition of Magic Cubes, and I shall in conclusion say a few words on the extension of the theory to squares of even numbers.

For the sake of simplicity I shall confine myself to the cube made up of the natural numbers from 0 to 26, both inclusive; which may be derived from the formula

$$x + 3y + 3^2z,$$

by giving successively to xyz the values 0.1.2 respectively.

The numbers represented by the nine following columns properly arranged, *i.e.* the second line being placed behind the first and the third behind the second, will form a magic cube; except as regards the diagonals, to which I shall afterwards direct attention.

A	*B*	*C*
$x_0 + 3y_0 + 3^2z_0$	$x_1 + 3y_2 + 3^2z_1$	$x_2 + 3y_1 + 3^2z_2$
$x_1 + 3y_1 + 3^2z_1$	$x_2 + 3y_0 + 3^2z_2$	$x_0 + 3y_2 + 3^2z_0$
$x_2 + 3y_2 + 3^2z_2$	$x_0 + 3y_1 + 3^2z_0$	$x_1 + 3y_0 + 3^2z_1$

D	*E*	*F*
$x_1 + 3y_1 + 3^2z_2$	$x_2 + 3y_0 + 3^2z_0$	$x_0 + 3y_2 + 3^2z_1$
$x_2 + 3y_2 + 3^2z_0$	$x_0 + 3y_1 + 3^2z_1$	$x_1 + 3y_0 + 3^2z_2$
$x_0 + 3y_0 + 3^2z_1$	$x_1 + 3y_2 + 3^2z_2$	$x_2 + 3y_1 + 3^2z_0$

G	*H*	*K*
$x_2 + 3y_2 + 3^2z_1$	$x_0 + 3y_1 + 3^2z_2$	$x_1 + 3y_0 + 3^2z_0$
$x_0 + 3y_0 + 3^2z_2$	$x_1 + 3y_2 + 3^2z_0$	$x_2 + 3y_1 + 3^2z_1$
$x_1 + 3y_1 + 3^2z_0$	$x_2 + 3y_0 + 3^2z_1$	$x_0 + 3y_2 + 3^2z_2$

In the first place the same number never recurs, so that the above comprise *all* the numbers from 0 to 26; secondly, the sum of each vertical column

$$= (x_0 + x_1 + x_2) + 3(y_0 + y_1 + y_2) + 3^2(z_0 + z_1 + z_2)$$
$$= 3(1 + 3 + 3^2)$$
$$= 39 \text{ (three times the mean number).}$$

Thirdly, if we take any number in A, and the corresponding numbers in B and C, we obtain the same result for the sum of the three; and similarly of the columns DEF, GHK, respectively.

Lastly, if we take any number in A and add to it the corresponding numbers in D and G respectively, we again have the same result, and similarly of the columns BEH, CFK respectively.

The diagonals are four in number. One of them may be found by taking the first of A (A_1), the second of E (E_2), and the third of K (K_3), which gives the sum

$$(a) \qquad x_0 + x_0 + x_0 + 3(y_0 + y_1 + y_2) + 3^2(z_0 + z_1 + z_2),$$

which, but for our having x_0 three times repeated instead of $x_0 + x_1 + x_2$, would likewise $= 39$. This defect may be remedied however by interchanging *throughout*, as we are at full liberty to do, x_0 and x_1; in which case we should have in (a) $(x_1 + x_1 + x_1)$ instead of $(x_0 + x_0 + x_0)$: and since

$$x_1 + x_1 + x_1 = x_0 + x_1 + x_2,$$

it is obvious that this diagonal will have the same value as any column of the square.

Another diagonal is $A_3 + E_2 + K_1$

$$= (x_2 + x_0 + x_1) + 3(y_2 + y_1 + y_0) + 3^2(z_2 + z_1 + z_0)$$
$$= 39.$$

A third is $C_1 + E_2 + G_3$

$$= (x_2 + x_0 + x_1) + 3(y_1 + y_1 + y_1) + 3^2(z_2 + z_1 + z_0)$$
$$= 39.$$

The last is $C_3 + E_2 + G_1$

$$= (x_1 + x_0 + x_2) + 3(y_0 + y_1 + y_2) + 3^2(z_1 + z_0 + z_2)$$
$$= 39.$$

Hence, as it will be shewn that the whole system may be formed by an invariable method from the first number of the first column, it follows that by properly choosing that number we shall always obtain a perfect cube.

I next come to the principle of formation. It will be seen that the numbers in any column are formed successively

from each other according to the order of their indices: thus in the column A the indices of x are 0.1.2, in D they are 1.2.0, but though the initial figure is different, the figures occur *in the same order;* and the same holds of the other columns.

Again, B is formed from A, as regards the x's and z's, by rejecting the head of each column and throwing it to the base: the column of y's is formed by elevating the base to the top: C is formed from B in the same manner as B from A, and E, F: H, K are formed from D and G respectively in the same manner as B and C are formed from A.

D is formed from A by treating the x's and y's in exactly the same way as the x's and z's are treated in the formation of B from A, and treating the z's in the former case as y in the latter; which we may, if we choose, express by saying that y and z are to be interchanged: but in practice it is better not to adopt this latter view.

G is formed from D in the same way as D from A; and E H, F K respectively might be formed from B and C respectively in a similar manner.

With regard to the number of different cubes to be obtained by the above method, I would observe that in the above example we may interchange at pleasure z_0 and z, z_1 and z_2, z_2 and z_0, so that if n be the number of cubes we should obtain independently of this consideration, we may by means of it increase the number to $3n$. Also we may interchange y_0 and y_2 (but not $y_1 \cdot y_2 : y_1 \cdot y_0$, as is easily seen from the consideration of the diagonals), and we may interchange x_0 and x_2, so that on the whole we shall obtain 2.2.3 cubes. But it is evident that in the above method z is, so to speak, the centre of the system, and as x and y have each equal claims in that respect the entire number of different cubes is $2^2.3^2$ or $1^2.2^2.3^2$.

It will be perceived that any one vertical row may be changed for another: thus we may interchange ADG with CFH, and so on; and in like manner any one horizontal row may be changed for another. The only effect of these changes will be on the diagonals of the cube, which however may always be adjusted by properly assuming the initial column.

The next remark I shall make bears on the subject of magic squares, as well as on that of magic cubes. The above method applies independently of the absolute values of $x_0y_0z_0$, $x_1y_1z_1$, $x_2y_2z_2$, provided only that, as regards one class of those quantities, the x's for example, one of the three values $x_0x_1x_2$ is the mean of the other two. Hence any series

of numbers which can be formed by giving to each of the quantities xyz respectively in the formula

$$x + 3y + 3^2z,$$

any three values whatever, consistently with the restriction that one of the three values of one of those variables is the mean of the other two, may be formed with a magic cube. I believe that in a cube containing $5^3 . 7^3$, &c. places, no restriction whatever is necessary, as to the values of any of the variables, beyond this, that they must not exceed 5.7, &c. in number respectively. A similar remark applies to magic squares, and thus forms a generalization of that part of the theory which I believe has not hitherto been adverted to.

If, instead of $x + 3y + 3^2z$, the series of numbers be represented by the formula

$$x + my + nz,$$

the same reasoning applies.

It is obvious that the theory may be extended, analytically, beyond the case of cube numbers. It might also not be impossible to contrive rectangles having the sums of their sides in a given ratio.

Instead of entering into any detailed account of the application of the above method to the case of squares of an even number of places, I shall subjoin two such squares, taken from an old French work, expressed according to the above system, which I analysed with a view to following up the theory; an intention however which, partly from lack of opportunity and partly inclination, I have not been able to carry out. To any person desirous of entering more fully into the subject, they may not be unserviceable.

The following is a magic square, composed of the natural numbers from 0 to 35 both inclusive. For the sake of convenience I put down only the indices of x and y,

(1)	4 + 4	(2)	5 + 1	(3)	3 + 4
	1 + 5		0 + 5		2 + 0
	4 + 3		5 + 2		2 + 2
	1 + 2		5 + 3		2 + 3
	1 + 0		0 + 0		3 + 5
	4 + 1		0 + 4		3 + 1
(4)	2 + 1	(5)	0 + 1	(6)	1 + 4
	3 + 0		5 + 5		4 + 0
	3 + 2		0 + 3		1 + 3
	3 + 3		0 + 2		4 + 2
	2 + 5		5 + 0		4 + 5
	2 + 4		5 + 4		1 + 1.

The next is composed of the numbers from 0 to 63 inclusive.

(1)	(2)	(3)	(4)
0 + 6	5 + 1	4 + 6	1 + 1
7 + 2	2 + 5	3 + 2	6 + 5
0 + 4	5 + 3	4 + 4	1 + 3
7 + 0	2 + 7	3 + 0	6 + 7
7 + 7	2 + 0	3 + 7	6 + 0
1 + 3	5 + 4	4 + 3	1 + 4
7 + 5	2 + 2	3 + 5	6 + 2
1 + 1	5 + 6	4 + 1	1 + 6

(5)	(6)	(7)	(8)
6 + 1	3 + 6	2 + 1	7 + 6
1 + 5	4 + 2	5 + 5	0 + 2
6 + 3	3 + 4	2 + 3	7 + 4
1 + 7	4 + 0	5 + 7	0 + 0
1 + 0	4 + 7	5 + 0	0 + 7
6 + 4	3 + 3	2 + 4	7 + 3
1 + 2	4 + 5	5 + 2	0 + 5
6 + 6	3 + 1	2 + 6	7 + 1.

I may observe that in my previous paper the number of magic squares, produced by the methods there indicated, is considerably under-estimated. It is there stated that in the case of a square of twenty-five places the effect produced by rejecting the *three* first x's from the top of the column would be to give the same result as would be obtained by rejecting the *two* first, but in the reverse order, which is not the fact. The squares obtained on the former principle of formation are distinct from those composed in the latter. Also the number of squares may be increased by combining the principle of the two methods explained in the paper alluded to. Thus the column of x's may be formed by rejecting the first, and that of the y's by rejecting the two first members (the initial x being always the mean value), and *vice versâ*.

Liverpool, December 29, 1845.

ON THE GEOMETRICAL REPRESENTATION OF THE MOTION OF A SOLID BODY.

By Arthur Cayley.

Let $P, Q, R, \ldots.$ be consecutive generating lines of a skew surface, and on these take points p', p ; q', q ; r', r .... such that $pq', qr' \ldots.$ are the shortest distances between P and Q, Q and R, &c. Then for the generating line P, the ratio of

incl^n of line P to plane Q q' p or of line Q to plane P p q'

the inclination of the lines *P, Q* to the distance *pq'* is said to be "the torsion," the angle *q'pq* is said to be the deviation, and the ratio of the inclination of the planes *Qpq'* and *Qqr'* to the inclination of *P* and *Q* is said to be the "skew curvature." And similarly for any other generating line; so that the torsion and deviation depend on the position of the consecutive line, and the skew curvature on the position of the two consecutive lines. The curve *pqr*. . . . is said to be the minimum distance curve. [When the skew surface degenerates into a developable surface, the torsion is infinite, the deviation a right angle, the skew curvature proportional to the curvature of the principal section, *i.e.* it is the distance of a point from the edge of regression, multiplied into the reciprocal of the radius of curvature, a product which is evidently constant along a generating line. Also the curve of minimum distance becomes the edge of regression.] A skew surface, considered independently of its position in space, is determined when for each generating line we know the torsion, deviation, and skew curvature. For, assuming arbitrarily the line *P* and the point *p*, also the plane in which *pq'* lies, the position of *Q* is completely determined from the given torsion and deviation; and then *Q* being known, the position of *R* is completely determined from the skew curvature for *P*, and the torsion and deviation for *Q*; and similarly the consecutive generating lines are to be determined.

Two skew surfaces are said to be "deformations" of each other, when for generating corresponding lines the torsion is always the same. Thus ~~a surface~~ will be deformed if considering the elements between the successive generating lines *P, Q* . . . . as rigid, these elements be made to revolve round the successive generating lines *P, Q* . . . . and to slide along them. [They are transformations, when not only the torsions but also the deviations are equal at corresponding generating lines: thus, if the sliding of the elements along *P, Q* . . . . be omitted, the new surface will be, not a deformation, but a transformation of the other.] No two skew surfaces can be made to roll and slide one upon the other, so that their successive generating lines coincide, unless one of them is a deformation of the other: and when this is the case, the rolling and sliding motions are *completely determined.* In fact the angular velocity of the generating line is the angular velocity round this line, into the difference of the skew curvatures of the two surfaces; the velocity of translation of the generating line in its own direction is to the angular velocity of the generating line, as the difference of the deviations is to

Tr

important

the torsion. [This includes also the case in which one surface is a transformation of the other, where the motion is evidently a rolling one.] A skew surface moving in this manner upon another of which it is the deformation, may be said to "glide" upon it. We may now state the kinematical theorem.

"Any motion whatever of a solid body in space may be represented as the 'gliding' motion of one skew surface upon another fixed in space, and of which it is ~~the~~ deformation." a/

A theorem which is to be considered as the generalization of the well known one—

"Any motion of a solid body round a fixed point may be represented as the rolling motion of a conical surface upon a second conic surface fixed in space."

And of the supplementary theorem—

"The angular velocity round the line of contact (the instantaneous axis) is to the angular velocity of this line as the difference of curvatures of the two cones at any point in the same line, to the reciprocal of the distance of the point from the vertex."

The analytical demonstration of this last theorem is rather interesting: it depends on the following formulæ. Forming two determinants, the first with the angular velocities round three axes fixed in space, and the first and second derived coefficients with respect to the time of these velocities; the other in the same way with the angular velocities round axes fixed in the body; the difference of these determinants is equal to the fourth power of the angular velocity into the square of the angular velocity of the instantaneous axis.

To show this, let p, q, r be the angular velocities round the axes fixed in the body; u, v, w those round axes fixed in space; ω the angular velocity round the instantaneous axis; ∇, Ω the two determinants: the theorem comes to

$$\nabla - \Omega = M,$$

where $M = \omega^2 (p'^2 + q'^2 + r'^2 - \omega'^2)$, or $\omega^2 (u'^2 + v'^2 + w'^2 - \omega'^2)$.

Here
$$\begin{aligned} u &= \alpha p + \beta q + \gamma r, \\ v &= \alpha' p + \beta' q + \gamma' r, \\ w &= \alpha'' p + \beta'' q + \gamma'' r. \end{aligned}$$

Whence
$$\begin{aligned} u' &= \alpha p' + \beta q' + \gamma r', \\ v' &= \alpha' p' + \beta' q' + \gamma' r', \\ w' &= \alpha'' p' + \beta'' q' + \gamma'' r', \end{aligned}$$

$\alpha = \beta'\gamma'' - \gamma'\beta''$ see Gregory p 51

(the remaining terms vanishing as is well known); and therefore by differentiation

$$vw' - v'w = \alpha\,(qr' - q'r) + \beta\,(rp' - r'p) + \gamma\,(pq' - p'q),$$
$$wu' - w'u = \alpha'\,(qr' - q'r) + \beta'\,(rp' - r'p) + \gamma'\,(pq' - p'q),$$
$$uv' - u'v = \alpha''\,(qr' - q'r) + \beta''\,(rp' - r'p) + \gamma''\,(pq' - p'q).$$

And hence

$$vw'' - v''w = \alpha\,(qr'' - q''r) + \beta\,(rp'' - r''p) + \gamma\,(pq'' - p''q) + \underline{u'\omega^2 - u\omega\omega'},$$
$$wu'' - w''u = \alpha'\,(qr'' - q''r) + \beta'\,(rp'' - r''p) + \gamma'\,(pq'' - p''q) + v'\omega^2 - v\omega\omega',$$
$$uv'' - u''v = \alpha''(qr'' - q''r) + \beta''(rp'' - r''p) + \gamma''(pq'' - p''q) + w'\omega^2 - w\omega\omega'.$$

how is this got?

And multiplying these by u', v', w', and adding, the required equation is immediately obtained.

In fact, if r be the distance of a point in the instantaneous axis from the vertex, and ρ, σ the radii of curvature of the two cones at that point, then

$$\frac{r}{\rho} = \frac{\omega^3}{M^{\frac{3}{2}}}\,\Omega, \quad \frac{r}{\sigma} = \frac{\omega^3}{M^{\frac{3}{2}}}\,\nabla.$$

As may be shown without difficulty, and the angular velocity of the instantaneous axis is given by the equation $\varpi = \dfrac{M^{\frac{1}{2}}}{\omega^2}$, whence the relation between the two angular velocities is

$$\omega : \varpi = \frac{1}{\rho} - \frac{1}{\sigma} : \frac{1}{r}.$$

ON THE ROTATION OF A SOLID BODY ROUND A FIXED POINT.

By Arthur Cayley.

The difficulty of completing elegantly the solution of this problem, in the case where no forces act upon the body, arises from the complexity and want of symmetry of the ordinary formulæ for determining the position of one set of rectangular axes with respect to another set; in consequence of which it has hitherto been considered necessary to make a particular supposition relative to the position of the fixed axes in space, viz. that one of them shall be perpendicular to the "invariable plane" of the rotating body. But some formulæ for the above purpose, given also by Euler, are entirely free from

these objections. Imagine two sets of axes Ax, Ay, Az, $Ax_{,}$, $Ay_{,}$, $Az_{,}$. The former set can be made to coincide with the second set, by a rotation θ round a certain axis AR, inclined to Ax, Ay, Az at angles f, g, h. (As usual f, g, h are the angles RAx, RAy, RAz considered as positive, and the rotation is in the same direction as a rotation round Az from x towards y). This axis may be termed the resultant axis, and the angle θ the resultant rotation. The formulæ of Euler express the coefficients of the transformation in terms of the resultant rotation and of the position of the resultant axis, *i.e.* in terms of θ and of the angles f, g, h, whose cosines are connected by the equation

$$\cos^2 f + \cos^2 g + \cos^2 h = 1.$$

This idea was improved upon by M. Rodrigues (Liouv. tom. v. p. 404), who introduced the quantities

$$\tan \tfrac{1}{2}\theta \cos f, \quad \tan \tfrac{1}{2}\theta \cos g, \quad \tan \tfrac{1}{2}\theta \cos h,$$

(quantities which will be represented by λ, μ, ν,) by means of which he expressed the coefficients as fractions, the numerators of which are very simple rational functions of the second order of λ, μ, ν, and which have the common denominator $(1 + \lambda^2 + \mu^2 + \nu^2)$. These quantities may conveniently be termed the "coordinates of the resultant rotation," and the denominator or the square of the secant of the semiangle of resultant rotation will be the "modulus" of the rotation. The elegance of these results led me to apply them to the mechanical question, and I gave in the *Journal* (vol. III. p. 224) the differential equations of motion obtained in terms of λ, μ, ν: which I integrated as in the common theory, by supposing one of the fixed axes to be perpendicular to the invariable plane. Though my attention was again called to the subject, by the connexion of some of these formulæ with Sir William Hamilton's theory of quaternions, no other way of performing the integration occurred to me. The grand discovery however of Jacobi, of the possibility of reducing to quadratures the two final differential equations of any mechanical problem, when the remaining integrals are known, induced me to resume the problem, and at least attempt to bring it so far as to obtain a differential equation of the first order between two variables only, the multiplier of which could be obtained theoretically by Jacobi's discovery. The choice of two new variables to which the equations of the problem led me, enabled me to effect this with the greatest simplicity; and the differential equation which I finally obtained, turned out

to be integrable *per se*, so that the laborious process of finding the multiplier became unnecessary. The new variables Ω, v have the following geometrical interpretations, $\Omega = k \tan \frac{1}{2} \theta \cos I$, where k is the principal moment, θ as before the angle of resultant rotation, and I is the inclination of the resultant axis to the perpendicular upon the invariable plane, and $v = k^2 \cos^2 \frac{1}{2} J$; where, if we imagine a line AQ having the same position relatively to the axes in fixed space, that the perpendicular upon the invariable plane has to the principal axes of the rotating body, then J is the inclination of this line to the above perpendicular. To the choice of these variables I was led by the analysis only. It will be seen that p, q, r are functions of v only, while λ, μ, ν contain besides the variable Ω. In obtaining these relations a singular equation $\Omega^2 = kv - k^2$ occurs (equation 13), which may also be written $1 + \tan^2 \frac{1}{2} \theta \cos^2 I = \sec^2 \frac{1}{2} \theta \cos^2 \frac{1}{2} J$, in which form the interpretation of the quantities I, J has just been given. The equation (17), it may be remarked, is self-evident: it expresses that the inclination of the resultant axis to the normal of the invariable plane, is equal to the inclination of the same axis to the line AQ. Now the resultant axis having the same inclination to the axes fixed in space as it has to the principal axes, and the line AQ the same inclinations to these fixed axes that the normal to the invariable plane has to the principal axes, the truth of the proposition becomes manifest. The correspondence in form between the systems (10) and (14) is also worth remarking. The final results at which I arrive are, that the time and the arc whose tangent is $\Omega \div k$, are each of them expressible as the integrals of certain algebraical functions of v. The notation throughout is the same as that made use of in the paper already quoted.

The equations of rotatory motion are

$$dt = \frac{dp}{P} = \frac{dq}{Q} = \frac{dr}{R} = \frac{d\lambda}{\Lambda} = \frac{d\mu}{\mathrm{M}} = \frac{d\nu}{\mathrm{N}} \quad \ldots\ldots\ldots (1),$$

where

$$\left.\begin{aligned} P &= \frac{1}{A}\left[(B-C)qr + \tfrac{1}{2}\left\{(1+\lambda^2)\frac{dV}{d\lambda} + (\lambda\mu+\nu)\frac{dV}{d\mu} + (\lambda\nu-\mu)\frac{dV}{d\nu}\right\}\right] \\ Q &= \frac{1}{B}\left[(C-A)rp + \tfrac{1}{2}\left\{(\mu\lambda-\nu)\frac{dV}{d\lambda} + (1+\mu^2)\frac{dV}{d\mu} + (\mu\nu+\lambda)\frac{dV}{d\nu}\right\}\right] \\ R &= \frac{1}{C}\left[(A-B)pq + \tfrac{1}{2}\left\{(\nu\lambda+\mu)\frac{dV}{d\lambda} + (\mu\nu-\lambda)\frac{dV}{d\mu} + (1+\nu^2)\frac{dV}{d\nu}\right\}\right] \end{aligned}\right\} \quad \ldots\ldots\ldots (2).$$

$$\left.\begin{aligned} \Lambda &= \tfrac{1}{2}\left\{(1+\lambda^2)p + (\lambda\mu-\nu)q + (\lambda\nu+\mu)r\right\} \\ \mathrm{M} &= \tfrac{1}{2}\left\{(\mu\lambda+\nu)p + (1+\mu^2)q + (\mu\nu-\lambda)r\right\} \\ \mathrm{N} &= \tfrac{1}{2}\left\{(\nu\lambda-\mu)p + (\mu\nu+\lambda)q + (1+\nu^2)r\right\} \end{aligned}\right\}\ldots\ldots(3).$$

And in the case where the forces vanish, the first three equations become simply

$$\left.\begin{aligned} P &= \frac{1}{A}(B-C)\,qr, \\ Q &= \frac{1}{B}(C-A)\,rp, \\ R &= \frac{1}{C}(A-B)\,pq. \end{aligned}\right\}\ldots\ldots\ldots\ldots(4).$$

In which case the usual four integrals of the system are

$$Ap^2 + Bq^2 + Cr^2 = h \ldots\ldots\ldots\ldots(5),$$

$$\left.\begin{aligned} Ap(1+\lambda^2-\mu^2-\nu^2) + 2Bq(\lambda\mu-\nu) + 2Cr(\nu\lambda+\mu) &= \mathrm{a}(1+\lambda^2+\mu^2+\nu^2) \\ 2Ap(\lambda\mu+\nu) + Bq(1+\mu^2-\nu^2-\lambda^2) + 2Cr(\mu\nu-\lambda) &= \mathrm{b}(1+\lambda^2+\mu^2+\nu^2) \\ 2Ap(\nu\lambda-\mu) + 2Bq(\mu\nu+\lambda) + Cr(1+\nu^2-\lambda^2-\mu^2) &= \mathrm{c}(1+\lambda^2+\mu^2+\nu^2) \end{aligned}\right\}\ldots\ldots(6).$$

Or as they may also be written,

$$\left.\begin{aligned} \mathrm{a}(1+\lambda^2-\mu^2-\nu^2) + 2\mathrm{b}(\lambda\mu+\nu) + 2\mathrm{c}(\nu\lambda-\mu) &= Ap(1+\lambda^2+\mu^2+\nu^2) \\ 2\mathrm{a}(\lambda\mu-\nu) + \mathrm{b}(1+\mu^2-\nu^2-\lambda^2) + 2\mathrm{c}(\mu\nu+\lambda) &= Bq(1+\lambda^2+\mu^2+\nu^2) \\ 2\mathrm{a}(\nu\lambda+\mu) + 2\mathrm{b}(\mu\nu-\lambda) + \mathrm{c}(1+\nu^2-\lambda^2-\mu^2) &= Cr(1+\lambda^2+\mu^2+\nu^2) \end{aligned}\right\}\ldots\ldots(6\ bis).$$

To which we may add,

$$A^2p^2 + B^2q^2 + C^2r^2 = k^2 \ldots\ldots\ldots\ldots(7);$$

where $$k^2 = \mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2 \ldots\ldots\ldots\ldots(8).$$

Introducing the quantities κ, Ω, (the former of which has been already made use of) given by the equations

$$\left.\begin{aligned} \kappa &= 1 + \lambda^2 + \mu^2 + \nu^2, \\ \Omega &= \lambda Ap + \mu Bq + \nu Cr \end{aligned}\right\}\ldots\ldots\ldots\ldots(9).$$

The equations (6) may be written under the form

$$\left.\begin{aligned} 2\lambda\Omega + 2\mu Cr - 2\nu Bq &= \kappa\,(Ap + \mathrm{a}) - 2Ap \\ -\,2\lambda Cr + 2\mu\Omega + 2\nu Ap &= \kappa\,(Bq + \mathrm{b}) - 2Bq \\ 2\lambda Bq - 2\mu Ap + 2\nu\Omega &= \kappa\,(Cr + \mathrm{c}) - 2Cr \end{aligned}\right\}\ldots\ldots(10).$$

Whence also, multiplying by Ap, Bq, Cr, and adding,

$$2\Omega^2 = \kappa\left\{k^2 + (Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c})\right\} - 2k^2 \ldots\ldots(11),$$

or writing $\qquad k^2 + (Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c}) = 2v \ldots\ldots (12),$

this becomes $\qquad \Omega^2 = kv - k^2 \ldots\ldots\ldots\ldots\ldots (13);$

an equation, the geometrical interpretation of which has already been given.

From the equations (10) we deduce the inverse system

$$\left.\begin{aligned} \mathrm{a}\Omega - \mathrm{b}Cr + \mathrm{c}Bq &= 2\lambda v - \Omega Ap \\ \mathrm{a}Cr + \mathrm{b}\Omega - \mathrm{c}Ap &= 2\mu v - \Omega Bq \\ -\mathrm{a}Bq + \mathrm{b}Ap + \mathrm{c}\Omega &= 2\nu v - \Omega Cr \end{aligned}\right\} \ldots\ldots (14),$$

which are easily verified by multiplying by Ω, Cr, $-Bq$; or by $-Cr$, Ω, Ap; or Bq, $-Ap$, Ω: adding and reducing, by which means the equations (10) are re-obtained. Hence also, if for shortness

$$\left.\begin{aligned} \Phi &= \mathrm{a}p + \mathrm{b}q + \mathrm{c}r \\ \nabla &= \mathrm{a}qr\,(B - C) + \mathrm{b}rp\,(C - A) + \mathrm{c}pq\,(A - B) \end{aligned}\right\} \ldots\ldots (15),$$

we have, multiplying by p, q, r, and adding,

$$\Omega\Phi - \nabla = 2v\,(\lambda p + \mu q + \nu r) - \Omega h \ldots\ldots (16).$$

To which may be added the equation

$$\Omega = \mathrm{a}\lambda + \mathrm{b}\mu + \mathrm{c}\nu \ldots\ldots\ldots\ldots\ldots (17),$$

which follows immediately from either of the systems (10) or (14).

We may also put the equations (10) under this other form,

$$\left.\begin{aligned} 2\lambda\Omega - 2\mu\mathrm{c} + 2\nu\mathrm{b} &= \kappa\,(Ap + \mathrm{a}) - 2\mathrm{a} \\ -2\lambda\mathrm{c} + 2\mu\Omega - 2\nu\mathrm{a} &= \kappa\,(Bq + \mathrm{b}) - 2\mathrm{b} \\ -2\lambda\mathrm{b} + 2\mu\mathrm{a} + 2\nu\Omega &= \kappa\,(Cr + \mathrm{c}) - 2\mathrm{c} \end{aligned}\right\} \ldots\ldots (10\ bis).$$

It may be remarked now, that p, q, r are functions of v; since we have to determine these quantities, the three equations

$$\left.\begin{aligned} Ap^2 + Bq^2 + Cr^2 &= h, \\ A^2p^2 + B^2q^2 + C^2r^2 &= k^2, \\ Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c} &= 2v - k^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots (18).$$

Also λ, μ, ν are given by the equations (14) as functions of p, q, r, Ω, *i.e.* of v, Ω. So that every thing is prepared for the investigation of the differential equation between v, Ω. To find this we have immediately

$$dv = \tfrac{1}{2}\,(A\mathrm{a}dp + B\mathrm{b}dq + C\mathrm{c}dr) = \tfrac{1}{2}\,\nabla dt \ldots\ldots (19),$$

from the equations (4) and (15). ∇ is of course to be considered as a given function of v. Again,

$$\Omega d\Omega = \tfrac{1}{2}\,(\kappa dv + v d\kappa) \ldots\ldots\ldots\ldots (20),$$

where $\qquad d\kappa = 2\,(\lambda d\lambda + \mu d\mu + \nu d\nu) \ldots\ldots\ldots\ldots (21);$

or from the equations (1), (3),

$$d\kappa = \kappa\,(\lambda p + \mu q + \nu r)\,dt \quad\ldots\ldots\ldots\ldots (22).$$

Whence, from (16),

$$2\upsilon d\kappa = \kappa\,\{\Omega\,(h + \Phi) - \nabla\}\,dt \quad\ldots\ldots\ldots (23);$$

or

$$2\,(\upsilon d\kappa + \kappa d\upsilon) = \kappa\Omega\,(h + \Phi)\,dt \quad\ldots\ldots\ldots (24).$$

Whence

$$d\Omega = \tfrac{1}{4}\,\kappa\,(h + \Phi)\,dt$$

$$= \tfrac{1}{4}\,\frac{\Omega^2 + k^2}{\upsilon}\,(h + \Phi)\,dt \quad\ldots\ldots\ldots (25).$$

And therefore, from (19),

$$\frac{2d\Omega}{\Omega^2 + k^2} = \frac{h + \Phi}{\upsilon\nabla}\,d\upsilon \quad\ldots\ldots\ldots\ldots (26),$$

the required differential equation, in which Φ, ∇ are given functions of (υ), *i.e.* they are functions of p, q, r by the equations (15), and these quantities are functions of υ by (18). The variables in (26) are therefore separated, and we have the integral equation

$$2\tan^{-1}\frac{\Omega}{k} = \delta + k\int\frac{(h + \Phi)\,d\upsilon}{\upsilon\nabla} \quad\ldots\ldots (27),$$

where δ is the constant of integration. The equation (19) gives also

$$t - \epsilon = 2\int\frac{d\upsilon}{\nabla} \quad\ldots\ldots\ldots\ldots\ldots (28);$$

and thus the solution of the problem is completely effected. The integrals may be taken from any particular value υ_0 of υ. The variable Ω may be exhibited as the integral of an *explicit* algebraical function, by recurring to the variable ϕ of the paper quoted.

Thus if

$$Ap_0^2 + Bq_0^2 + Cr_0^2 = h,$$

$$A^2p_0^2 + B^2q_0^2 + C^2r_0^2 = k^2,$$

$$Ap_0\mathrm{a} + Bq_0\mathrm{b} + Cr_0\mathrm{c} = 2\upsilon_0 - k^2;$$

then

$$\sqrt{\left\{p_0^2 - \frac{1}{A}\,(C - B)\,\phi\right\}},\quad \sqrt{\left\{q_0^2 - \frac{1}{A}\,(A - C)\,\phi\right\}}$$

$$\sqrt{\left\{r_0^2 - \frac{1}{C}\,(B - A)\,\phi\right\}},$$

$$dt = \tfrac{1}{2}\,\frac{d\phi}{pqr} = \frac{2d\upsilon}{\nabla},\quad\text{or}\quad \frac{d\upsilon}{\nabla} = \tfrac{1}{4}\,\frac{d\phi}{pqr};$$

Formulæ for the variation of the arbitrary constants, in the case of any distributing forces acting upon the body, will be given in a subsequent paper.

$$\text{whence } 4\tan^{-1}\frac{\Omega}{k} = 2\delta + k\int_0 \frac{(h + \mathrm{a}p + \mathrm{b}q + cr)\,d\phi}{(k^2 + Ap\mathrm{a} + Bq\mathrm{b} + Cr\mathrm{c})\,pqr}.$$

In which form it is exactly analogous to the equation there obtained, p. 230,

$$4\tan^{-1}\nu_0 = \int\frac{(h + kr)\,d\phi}{(k + Cr)\,pqr}.$$

(*To be continued.*)

ON THE LAWS OF EQUILIBRIUM AND MOTION OF SOLID AND FLUID BODIES.

By Samuel Haughton, Fellow of Trinity College, Dublin.

A general investigation of the laws of equilibrium and motion of solid and fluid bodies must consist essentially of two parts: the investigation of the differential equations of equilibrium and motion, and the subsequent integration of these equations. In this paper I propose to treat of both these subjects. The principles from which I set out are extremely simple and do not involve any assumptions, except such as will readily appear to be natural consequences of our conception of the nature of the bodies which surround us, and the results to which the investigation leads are in accordance with all the known mathematical laws of solid and fluid bodies. Some of these results are also, so far as I am aware, new, and seem to throw light on the difficult problem of the equilibrium and motion of solid elastic bodies.

The *method* which I have followed is that of the '*Mécanique Analytique*' of Lagrange, of which such a successful application has been made by Professor McCullagh to the investigation of the mechanical laws of light, and consists in the application of the Calculus of Variations to the principles of rational mechanics. One great difficulty of this method arises from its comprehensiveness, and the labour of the mathematician is more frequently to *distinguish* between different cases included in the formulæ, than to show that these cases are all *contained* in them. Previously, therefore, to entering upon the investigation of the differential equations, I shall endeavour to establish a distinction between solids and fluids; on which subject much confusion and contradiction seems to exist between the writers who have treated most of these subjects.

The most general conception of solids and fluids, is that of 'an immense assemblage of molecules separated from each

other by indefinitely small distances'; if we add to this general notion, the assertion, that '*these molecules act on each other only in the line joining them,*' we shall have a definition of the medium whose laws I propose to investigate. Suppose now that this medium is acted on by *no external* forces, and abandoned solely to the action of its molecules on each other; the distinction which I conceive to exist between solids and fluids is the following:

That in solid bodies the resultant of all the forces exerted by all the surrounding molecules on any molecule (m), *is zero. That in fluids, whether liquid or gaseous, this is not the case, and that consequently the fluid (no external pressures or forces acting) would be dissipated.*

That this is the correct distinction of these two classes of bodies, will, I hope, be made clear by the following investigation, and at present it will be sufficient to observe that it agrees with the ordinary notion of a fluid: in such a body we suppose a pressure (p) to exist at each point (x, y, z), which equilibrates the external forces, such as the forces arising from the sides of the vessel containing the fluid, &c. Hence, if the *external forces cease to act*, the pressure being transmitted to the external surface of the fluid would dissipate it.

The case of a fluid in a closed vessel is not the case here considered, for the pressures of the sides of the vessel are, in this case, the external forces, which, together with the molecular forces, produce equilibrium: but the molecular forces themselves are not, I conceive, in equilibrium. This is manifestly true of gases, and I consider the same to be true of liquids. The distinction between liquids and gases is, probably, *relative* to the ordinary external forces in action at the surface of the earth, such as gravity, the pressure of the atmosphere, &c.

I proceed, without further delay, to the laws of the medium, whether solid or fluid.

The general equation of equilibrium of the points composing a medium is

$$\iiint(X\delta\xi + Y\delta\eta + Z\delta\zeta)\,dm = \iiint\delta V\,dx\,dy\,dz \;\ldots\ldots\; (1),$$

in which equation the left-hand member is the sum of the '*moments*' of the external forces, and the right-hand member is the sum of the *moments* of the internal forces. I use the term *moments* as defined by Lagrange.

The function V depends, in general, on the particular nature of the medium considered, and its form must, in the present case, be deduced from the definition already given. x, y, z are the coordinates of the position of rest of any

molecule (m); and $x+\xi$, $y+\eta$, $z+\zeta$ are the coordinates of the same molecule when displaced by the action of external forces.

When the molecules are in a state of rest, not acted upon by any external forces, the force exerted by any molecule (m') on (m) will be, in general, a function of the *distance* (mm'), and of the *direction* of the line (mm'): if now, by means of external forces, the molecules $(m, m', \&c.)$ assume new positions, the force exerted by m' on m will be represented in general by the expression

$$f(\rho, \alpha, \beta, \gamma, \rho'),$$

$\rho, \alpha, \beta, \gamma$ being the *original* length and direction of the line (mm'), and ρ' being the alteration in ρ; ρ' being small, the function (f) may be represented by

$$f = F_0 + 2F_1\rho' + 3F_2\rho'^2 + \&c. \dots\dots\dots (2),$$

F_0, F_1, &c. being functions of $(\rho, \alpha, \beta, \gamma)$.

As the force (f) is a force tending to alter $\rho+\rho'$, its *moment* will be $f.\delta\rho'$, or neglecting ρ'^2, &c.

$$F_0\delta\rho' + 2F_1\rho'\delta\rho'.$$

Hence $$\delta V = \Sigma\,\{F_0\delta\rho' + F_1\delta(\rho'^2)\};$$

and therefore

$$V = \int_0^\infty\int_0^\pi\int_0^{2\pi}(F_0\rho' + F_1\rho'^2)\,\rho^2\sin\theta d\rho\, d\theta\, d\phi \dots . (3).$$

The value of (ρ'), to be substituted in this expression for V, is thus found. Let $x, y, z, x+a, y+b, z+c$, be the coordinates of rest of m and m'; then in the changed position, if x, y, z, become $x+\xi$, $y+\eta$, $z+\zeta$, the coordinates of m' will become

$$\left.\begin{array}{l} x+\xi+a+\dfrac{d\xi}{dx}a+\dfrac{d\xi}{dy}b+\dfrac{d\xi}{dz}c, \\[2ex] y+\eta+b+\dfrac{d\eta}{dx}a+\dfrac{d\eta}{dy}b+\dfrac{d\eta}{dz}c, \\[2ex] z+\zeta+c+\dfrac{d\zeta}{dx}a+\dfrac{d\zeta}{dy}b+\dfrac{d\zeta}{dz}c, \end{array}\right\}.$$

$\rho+\rho'$ is equal to the square root of the sum of the squares of the differences of the coordinates of its extreme points, *i.e.*

$$\rho+\rho' = \sqrt{\left\{\left(a+\frac{d\xi}{dx}a+\frac{d\xi}{dy}b+\frac{d\xi}{dz}c\right)^2+\left(b+\frac{d\eta}{dx}a+\frac{d\eta}{dy}b+\frac{d\eta}{dz}c\right)^2\right.}$$
$$\left.+\left(c+\frac{d\zeta}{dx}a+\frac{d\zeta}{dy}b+\frac{d\zeta}{dz}c\right)^2\right\};$$

and, neglecting the smaller quantities, we obtain

$$\rho + \rho' = \rho + \rho\left(\frac{d\xi}{dx}\cos^2 a + \frac{d\eta}{dy}\cos^2\beta + \frac{d\zeta}{dz}\cos^2\gamma\right.$$
$$\left. + u\cos\beta\cos\gamma + v\cos a\cos\gamma + w\cos a\cos\beta\right),$$

where
$$u = \frac{d\eta}{dz} + \frac{d\zeta}{dy},$$
$$v = \frac{d\zeta}{dx} + \frac{d\xi}{dz},$$
$$w = \frac{d\xi}{dy} + \frac{d\eta}{dx};$$

and also
$$\rho = \sqrt{(a^2 + b^2 + c^2)}.$$

Hence finally we obtain

$$\rho' = \rho\left\{\frac{d\xi}{dx}\cos^2 a + \frac{d\eta}{dy}\cos^2\beta + \frac{d\zeta}{dz}\cos^2\gamma + \left(\frac{d\eta}{dz} + \frac{d\zeta}{dy}\right)\cos\beta\cos\gamma\right.$$
$$\left. + \left(\frac{d\zeta}{dx} + \frac{d\xi}{dz}\right)\cos a\cos\gamma + \left(\frac{d\xi}{dy} + \frac{d\eta}{dx}\right)\cos a\cos\beta\right\}.$$

This is the value of ρ', to be substituted in V, which will then consist of two parts, V_0 and V_1, depending upon F_0 and F_1, V_0 being homogeneous and linear with respect to

$$\frac{d\xi}{dx},\ \frac{d\eta}{dy},\ \frac{d\zeta}{dz},\ u,\ v,\ w;$$

and V_1 homogeneous, and of the second order with respect to the same six quantities.

Returning now to the distinction drawn between solids and fluids, it will appear from that distinction that in fluids the function V will be $V_0 + V_1$, while for solids $V_0 = 0$: for if in (2) we suppose $\rho' = 0$, we shall have $F_0 = f$; hence the definition of a solid requires that the forces F_0 which correspond to the case of *no external forces* acting should equilibrate.

I shall now determine the equations of equilibrium arising from V_0, which does not vanish for fluids, and then examine the case of solids; for which $V_0 = 0$, and which depend only on V_1.

By formula (3), we have

$$V_0 = \int_0^\infty\int_0^\pi\int_0^{2\pi} F_0\rho'.\rho^2 \sin\theta\, d\rho\, d\theta\, d\phi.$$

Hence

$$V_0 = \int_0^\infty \int_0^\pi \int_0^{2\pi} F_0 \left(\frac{d\xi}{dx}\cos^2\alpha + \frac{d\eta}{dy}\cos^2\beta + \frac{d\zeta}{dz}\cos^2\gamma\right.$$
$$\left. + u\cos\beta\cos\gamma + v\cos\alpha\cos\gamma + w\cos\alpha\cos\beta\right)\rho^3 \sin\theta\, d\rho\, d\theta\, d\phi \ldots (4).$$

In this equation F_0 is a function of ρ only, since the medium is equally elastic in all directions, and

$$\cos\alpha = \cos\phi\sin\theta,$$
$$\cos\beta = \sin\phi\sin\theta,$$
$$\cos\gamma = \cos\theta.$$

The coefficients of $\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}$, u, v, w, are six triple integrals; and in the present case it is easily seen that the coefficients of u, v, w, are zero, and those of $\frac{d\xi}{dx}, \frac{d\eta}{dy}, \frac{d\zeta}{dz}$ equal to each other. Their common value is

$$p = \frac{4\pi}{3}\int_0^\infty F_0 . \rho^3 d\rho,$$

which is obtained by integrating twice, with respect to θ and ϕ.

Hence finally we obtain

$$V_0 = p\left(\frac{d\xi}{dx} + \frac{d\eta}{dy} + \frac{d\zeta}{dz}\right) \ldots\ldots\ldots\ldots (5).$$

By the general equation of equilibrium, we have

$$\iiint (X\delta\xi + Y\delta\eta + Z\delta\zeta)\, dm = \iiint \delta V_0\, dx\, dy\, dz.$$

Substituting the value of V_0, and proceeding according to the principles of the Calculus of Variations, we obtain

$$\iiint (X\delta\xi + Y\delta\eta + Z\delta\zeta)\, dm = \iint p\,\delta\xi\, dy\, dz + \iint p\,\delta\eta\, dx\, dz$$
$$+ \iint p\,\delta\zeta\, dx\, dy,$$
$$- \iiint \left(\frac{dp}{dx}\delta\xi + \frac{dp}{dy}\delta\eta + \frac{dp}{dz}\delta\zeta\right) dx\, dy\, dz \ldots\ldots (6),$$

the triple integrals giving the equations of equilibrium, and the double integrals giving the conditions at the limits.

Hence, if the density be expressed by ϵ, we shall have the expression $dm = \epsilon dxdydz$, and the equations of equilibrium will be

$$\left.\begin{aligned} -\epsilon X &= \frac{dp}{dx} \\ -\epsilon Y &= \frac{dp}{dy} \\ -\epsilon Z &= \frac{dp}{dz} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (7),$$

which are the well-known equations of Hydrostatics. Hence, if the forces (X, Y, Z) be zero at all points of the fluid, the quantity p must be constant; and vice versâ, if p be constant for all points of the fluid, the forces (X, Y, Z) must be zero—(this includes the case of homogeneous fluids): in such a case the function V_0 will give only the condition at the limits expressed by the double integrals (6); which is identical with that found by Lagrange, and expresses that there must be a normal pressure at every point of the bounding surface, constant and equal to p, in order that the fluid should remain in equilibrium. This condition at the limits is not necessary in solids, since for them V_0, and therefore p, is equal to zero.

It follows from the views I have adopted in this paper, that the ordinary equations of Hydrostatics and Hydrodynamics are only a first approximation to the whole equations, and that in some cases, particularly in Hydrodynamics, this approximation may be insufficient: in such cases we should add to the equations (7) the terms arising from V_1, which are common to solids and fluids.

In general $$V = V_0 + V_1 + V_2 + \&c.,$$

and the terms V_0, V_1, &c. will give rise, in the equations of equilibrium, to differential coefficients of the first, second, &c. order.

I shall now take up the general discussion of the equations of equilibrium and motion arising from the function (V_1).

From equation (3) we see that

$$V_1 = \int_0^\infty \int_0^\pi \int_0^{2\pi} F_1 . (\rho')^2 . \rho^2 \sin \theta d\rho d\theta d\phi.$$

Hence, making $d\omega = \rho^2 \sin \theta \, d\rho \, d\theta \, d\phi$, we shall have

$$V_1 = \int_0^\infty \int_0^\pi \int_0^{2\pi} F_1 \left(\frac{d\xi}{dx} \cos^2 \alpha + \frac{d\eta}{dy} \cos^2 \beta + \frac{d\zeta}{dz} \cos^2 \gamma \right.$$
$$\left. + u \cos \beta \cos \gamma + v \cos \alpha \cos \gamma + w \cos \alpha \cos \beta \right)^2 \rho^2 d\omega. \dots (8).$$

Therefore

$$2 V_1 = \left\{ A \left(\frac{d\xi}{dx} \right)^2 + B \left(\frac{d\eta}{dy} \right)^2 + C \left(\frac{d\zeta}{dz} \right)^2 \right\} + \{ Lu^2 + Mv^2 + Nw^2 \}$$
$$+ 2 \left(L \frac{d\eta}{dy} . \frac{d\zeta}{dz} + M \frac{d\xi}{dx} . \frac{d\zeta}{dz} + N \frac{d\xi}{dx} . \frac{d\eta}{dy} \right)$$
$$+ 2 (\alpha_1 vw + \beta_2 uw + \gamma_3 uv)$$

$$+ 2\left\{u\left(a_1 \frac{d\xi}{dx} + \beta_1 \frac{d\eta}{dy} + \gamma_1 \frac{d\zeta}{dz}\right) + v\left(a_2 \frac{d\xi}{dx} + \beta_2 \frac{d\eta}{dy} + \gamma_2 \frac{d\zeta}{dz}\right)\right.$$
$$\left. + w\left(a_3 \frac{d\xi}{dx} + \beta_3 \frac{d\eta}{dy} + \gamma_3 \frac{d\zeta}{dz}\right)\right\} \ldots (9).$$

In this value of V_1 there are 21 terms, but only 15 constants, since the coefficients of six of the terms are expressed by the same integrals as some other terms of the function.

The values of the 15 coefficients are given by the following expressions:

$$A = 2\iiint F_1 \cos^4 a . \rho^2 d\omega, \qquad B = 2\iiint F_1 \cos^4 \beta . \rho^2 d\omega,$$
$$C = 2\iiint F_1 \cos^4 \gamma . \rho^2 d\omega,$$
$$L = 2\iiint F_1 \cos^2 \beta \cos^2 \gamma . \rho^2 d\omega, \quad M = 2\iiint F_1 \cos^2 a \cos^2 \gamma . \rho^2 d\omega,$$
$$N = 2\iiint F_1 \cos^2 a \cos^2 \beta . \rho^2 d\omega.$$
$$a_1 = 2\iiint F_1 \cos^2 a \cos \beta \cos \gamma . \rho^2 d\omega, \quad \beta_1 = 2\iiint F_1 \cos^3 \beta \cos \gamma . \rho^2 d\omega,$$
$$\gamma_1 = 2\iiint F_1 \cos^3 \gamma \cos \beta . \rho^2 d\omega,$$
$$a_2 = 2\iiint F_1 \cos^3 a \cos \gamma . \rho^2 d\omega, \quad \beta_2 = 2\iiint F_1 \cos^2 \beta \cos a \cos \gamma . \rho^2 d\omega,$$
$$\gamma_2 = 2\iiint F_1 \cos^3 \gamma \cos a . \rho^2 d\omega,$$
$$a_3 = 2\iiint F_1 \cos^3 a \cos \beta . \rho^2 d\omega, \quad \beta_3 = 2\iiint F_1 \cos^3 \beta \cos a . \rho^2 d\omega,$$
$$\gamma_3 = 2\iiint F_1 \cos^2 \gamma \cos a \cos \beta . \rho^2 d\omega.$$

This value of V_1 must now be substituted in the equation

$$\iiint (X\delta\xi + Y\delta\eta + Z\delta\zeta)\, dm = \iiint \delta V_1 dxdydz,$$

which must then be treated by the rules of the Calculus of Variations.

$$(a)\ \tfrac{1}{2}\delta \iiint \left\{A\left(\frac{d\xi}{dx}\right)^2 + B\left(\frac{d\eta}{dy}\right)^2 + C\left(\frac{d\zeta}{dz}\right)^2\right\} dxdydz$$
$$= \iint A \frac{d\xi}{dx} \delta\xi dydz$$
$$+ \iint B \frac{d\eta}{dy} \delta\eta dxdz$$
$$+ \iint C \frac{d\zeta}{dz} \delta\zeta dxdy$$
$$- \iiint \left(A \frac{d^2\xi}{dx^2} \delta\xi + B \frac{d^2\eta}{dy^2} \delta\eta + C \frac{d^2\zeta}{dz^2} \delta\zeta\right) dxdydz.$$

$$(b)\ \tfrac{1}{2}\delta \iiint (Lu^2 + Mv^2 + Nw^2)\, dxdydz = \iint (Mv\delta\zeta + Nw\delta\eta)\, dydz$$
$$+ \iint (Nw\delta\xi + Lu\delta\zeta)\, dxdz$$
$$+ \iint (Lu\delta\eta + Mv\delta\xi)\, dxdy$$

$$-\iiint\left\{\left(M\frac{dv}{dz}+N\frac{dw}{dy}\right)\delta\xi+\left(N\frac{dw}{dx}+L\frac{du}{dz}\right)\delta\eta\right.$$
$$\left.+\left(L\frac{du}{dy}+M\frac{dv}{dx}\right)\delta\zeta\right\}dxdydz.$$

(*c*) $\frac{1}{2}\delta\iiint 2\left(L\frac{d\eta}{dy}\cdot\frac{d\zeta}{dz}+M\frac{d\xi}{dx}\cdot\frac{d\zeta}{dz}+N\frac{d\xi}{dx}\cdot\frac{d\eta}{dy}\right)dxdydz$

$$=\iint\left(M\frac{d\zeta}{dz}+N\frac{d\eta}{dy}\right)\delta\xi dydz$$
$$+\iint\left(N\frac{d\xi}{dx}+L\frac{d\zeta}{dz}\right)\delta\eta dxdz$$
$$+\iint\left(L\frac{d\eta}{dy}+M\frac{d\xi}{dx}\right)\delta\zeta dxdy$$
$$-\iiint\left\{\left(M\frac{d^2\zeta}{dxdz}+N\frac{d^2\eta}{dxdy}\right)\delta\xi+\left(N\frac{d^2\xi}{dxdy}+L\frac{d^2\zeta}{dydz}\right)\delta\eta\right.$$
$$\left.+\left(L\frac{d^2\eta}{dydz}+M\frac{d^2\xi}{dxdz}\right)\delta\zeta\right\}dxdydz.$$

(*d*) $\frac{1}{2}\delta\iiint 2(\alpha_1 vw+\beta_2 uw+\gamma_3 uv)\,dxdydz$

$$=\iint\{\alpha_1(v\delta\eta+w\delta\zeta)+u(\beta_2\delta\eta+\gamma_3\delta\zeta)\}\,dydz$$
$$+\iint\{\beta_2(w\delta\zeta+u\delta\xi)+v(\gamma_3\delta\zeta+\alpha_1\delta\xi)\}\,dxdz$$
$$+\iint\{\gamma_3(u\delta\xi+v\delta\eta)+w(\alpha_1\delta\xi+\beta_2\delta\eta)\}\,dxdy.$$
$$-\iiint\left\{\alpha_1\left(\frac{dv}{dy}+\frac{dw}{dz}\right)+\beta_2\frac{du}{dy}+\gamma_3\frac{du}{dz}\right\}\delta\xi\,dxdydz$$
$$-\iiint\left\{\beta_2\left(\frac{dw}{dz}+\frac{du}{dx}\right)+\gamma_3\frac{dv}{dz}+\alpha_1\frac{dv}{dx}\right\}\delta\eta\,dxdydz$$
$$-\iiint\left\{\gamma_3\left(\frac{du}{dx}+\frac{dv}{dy}\right)+\alpha_1\frac{dw}{dx}+\beta_2\frac{dw}{dy}\right\}\delta\zeta dxdydz.$$

If we assume

$$\aleph=\alpha_1\frac{d\xi}{dx}+\beta_1\frac{d\eta}{dy}+\gamma_1\frac{d\zeta}{dz},$$
$$\beth=\alpha_2\frac{d\xi}{dx}+\beta_2\frac{d\eta}{dy}+\gamma_2\frac{d\zeta}{dz},$$
$$\gimel=\alpha_3\frac{d\xi}{dx}+\beta_3\frac{d\eta}{dy}+\gamma_3\frac{d\zeta}{dz},$$

we shall have

$$(e)\ \tfrac{1}{2}\,\delta\iiint 2\,(u\aleph + v\beth + w\gimel)\,dxdydz = \iint(\beth\delta\zeta + \gimel\delta\eta)\,dydz$$
$$+ \iint(\gimel\delta\xi + \aleph\delta\zeta)\,dxdz$$
$$+ \iint(\aleph\delta\eta + \delta\xi)\,\beth dxdy$$
$$+ \iint(\alpha_1 u + \alpha_2 v + \alpha_3 w)\,\delta\xi\,dydz$$
$$+ \iint(\beta_1 u + \beta_2 v + \beta_3 w)\,\delta\eta\,dxdz$$
$$+ \iint(\gamma_1 u + \gamma_2 v + \gamma_3 w)\,\delta\zeta\,dxdy$$
$$- \iiint\left\{\left(\frac{d\beth}{dz} + \frac{d\gimel}{dy}\right)\delta\xi + \left(\frac{d\gimel}{dx} + \frac{d\aleph}{dz}\right)\delta\eta + \left(\frac{d\aleph}{dy} + \frac{d\beth}{dx}\right)\delta\zeta\right\} dxdydz$$
$$- \iiint\left\{\left(\alpha_1\,\frac{du}{dx} + \alpha_2\,\frac{dv}{dx} + \alpha_3\,\frac{dw}{dx}\right)\delta\xi\right.$$
$$+ \left(\beta_1\,\frac{du}{dy} + \beta_2\,\frac{dv}{dy} + \beta_3\,\frac{dw}{dy}\right)\delta\eta$$
$$\left. + \left(\gamma_1\,\frac{du}{dz} + \gamma_2\,\frac{dv}{dz} + \gamma_3\,\frac{dw}{dz}\right)\delta\zeta\right\} dxdydz.$$

The sum of these five quantities, added together, is the expression for the variation $\iiint\delta V_1 dxdydz$. Hence we shall have

$$\iiint\delta V_1 dxdydz = \Delta - \iiint(P\delta\xi + Q\delta\eta + R\delta\zeta)\,dxdydz\ldots(10),$$

where Δ represents the sum of all the double integrals, and will belong essentially to the limits, and the quantities P, Q, R give the differential equations of equilibrium and motion, and have the following values:

$$P = A\,\frac{d^2\xi}{dx^2} + N\frac{d^2\xi}{dy^2} + M\frac{d^2\xi}{dz^2} + 2\left(\alpha_1\,\frac{d^2\xi}{dydz} + \alpha_2\,\frac{d^2\xi}{dxdz} + \alpha_3\,\frac{d^2\xi}{dxdy}\right)$$
$$+ \alpha_3\,\frac{d^2\eta}{dx^2} + \beta_3\,\frac{d^2\eta}{dy^2} + \gamma_3\,\frac{d^2\eta}{dz^2} + 2\left(\alpha_1\,\frac{d^2\eta}{dxdz} + N\,\frac{d^2\eta}{dxdy} + \beta_2\,\frac{d^2\eta}{dydz}\right)$$
$$+ \alpha_2\,\frac{d^2\zeta}{dx^2} + \beta_2\,\frac{d^2\zeta}{dy^2} + \gamma_2\,\frac{d^2\zeta}{dz^2} + 2\left(\alpha_1\,\frac{d^2\zeta}{dxdy} + M\,\frac{d^2\zeta}{dxdz} + \gamma_3\,\frac{d^2\zeta}{dydz}\right).$$

$$Q = B\,\frac{d^2\eta}{dy^2} + L\,\frac{d^2\eta}{dz^2} + N\,\frac{d^2\eta}{dx^2} + 2\left(\beta_2\,\frac{d^2\eta}{dxdz} + \beta_3\,\frac{d^2\eta}{dxdy} + \beta_1\,\frac{d^2\eta}{dydz}\right)$$
$$+ \alpha_1\,\frac{d^2\zeta}{dx^2} + \beta_1\,\frac{d^2\zeta}{dy^2} + \gamma_1\,\frac{d^2\zeta}{dz^2} + 2\left(\beta_2\,\frac{d^2\zeta}{dxdy} + L\,\frac{d^2\zeta}{dydz} + \gamma_3\,\frac{d^2\zeta}{dxdz}\right)$$
$$+ \alpha_3\,\frac{d^2\xi}{dx^2} + \beta_3\,\frac{d^2\xi}{dy^2} + \gamma_3\,\frac{d^2\xi}{dz^2} + 2\left(\beta_2\,\frac{d^2\xi}{dydz} + N\,\frac{d^2\xi}{dxdy} + \alpha_1\,\frac{d^2\xi}{dxdz}\right).$$

$$R = C\frac{d^2\zeta}{dz^2} + M\frac{d^2\zeta}{dx^2} + L\frac{d^2\zeta}{dy^2} + 2\left(\gamma_3\frac{d^2\zeta}{dxdy} + \gamma_1\frac{d^2\zeta}{dydz} + \gamma_2\frac{d^2\zeta}{dxdz}\right)$$
$$+ \alpha_2\frac{d^2\xi}{dx^2} + \beta_2\frac{d^2\xi}{dy^2} + \gamma_2\frac{d^2\xi}{dz^2} + 2\left(\gamma_3\frac{d^2\xi}{dydz} + M\frac{d^2\xi}{dxdz} + \alpha_1\frac{d^2\xi}{dxdy}\right)$$
$$+ \alpha_1\frac{d^2\eta}{dx^2} + \beta_1\frac{d^2\eta}{dy^2} + \gamma_1\frac{d^2\eta}{dz^2} + 2\left(\gamma_3\frac{d^2\eta}{dxdz} + L\frac{d^2\eta}{dydz} + \beta_2\frac{d^2\eta}{dxdy}\right).$$

The general equation of equilibrium and motion, corresponding to the function V_1, will be

$$\iiint \varepsilon (X\delta\xi + Y\delta\eta + Z\delta\zeta)\, dxdydz$$
$$= \Delta - \iiint (P\delta\xi + Q\delta\eta + R\delta\zeta)\, dxdydz,$$

where $dm = \varepsilon dxdydz$; therefore the equations of equilibrium are

$$-\varepsilon X = P, \quad -\varepsilon Y = Q, \quad -\varepsilon Z = R. \ldots\ldots (11).$$

These equations are the equations of equilibrium of a solid body expressed in their most general form, without making any supposition as to the arrangement of the molecules in the body, and supposing the force in action between any two molecules to be a function, as well of the *direction* of the line joining them, as of the *length* of that line; which is the most general conception of a crystalline structure.

NOTE. As the continuation of this paper will not appear until the next No. of the *Cambridge and Dublin Mathematical Journal,* I shall here mention some of the results at which I have arrived. Having simplified the function V_1, I have integrated the equations of motion by means of a particular integral, which is general enough to give, by means of the relations among the constants, many geometrical properties of the motion of elastic solids. In examining the propagation of waves through the medium, I have used the *surface of wave-slowness,* which is of the sixth degree, and possesses nodes in its principal planes, which give rise to a theory of conical refraction of the vibrations of solids, somewhat analogous to the corresponding case of light. In the case of homogeneous, uncrystalline bodies, the whole theory becomes exceedingly simple.

Trinity College, Dublin, March, 1846.

(*To be continued.*)

ON A FORMULA FOR DETERMINING THE OPTICAL CONSTANTS OF DOUBLY REFRACTING CRYSTALS.

By G. G. Stokes, M.A., Fellow of Pembroke College.

In order to explain the object of this formula, it will be necessary to allude to the common method of determining the optical constants. Two plane faces of the crystal are selected, which are parallel to one of the axes of elasticity; or if such do not present themselves, they are obtained artificially by grinding. A pencil of light is transmitted across these faces in a plane perpendicular to them both, as in the case of an ordinary prism. This pencil is by refraction separated into two, of which one is polarized in the plane of incidence, and follows the ordinary law of refraction, while the other is polarized in a plane perpendicular to the plane of incidence, and follows a different law. It will be convenient to call these pencils respectively the *ordinary* and the *extraordinary*, in the case of biaxal, as well as uniaxal crystals. The minimum deviation of the ordinary pencil is then observed, and one of the optical constants, namely that which relates to the axis of elasticity parallel to the refracting edge, is thus determined by the same formula which applies to ordinary media. This formula will also give one of the other constants, by means of the observation of the minimum deviation of the extraordinary pencil, in the particular case in which one of the principal planes of the crystal bisects the angle between the refracting planes: but if this condition be not fulfilled it will be necessary to employ either two or three prisms, according as the crystal is uniaxal or biaxal, to determine all the constants. The extraordinary pencil, however, need not in any case be rejected, provided only a formula be obtained connecting the minimum deviation observed with the optical constants. It will thus be possible to determine all the constants with a smaller number of prisms; the necessity of using artificial faces may often be obviated; or if two faces are cut as nearly as may be equally inclined to one of the axes of elasticity lying in the plane of incidence, or one cut face is used with a natural face, the errors of cutting may be allowed for.

Let AEB be a section of the prism by the plane of refraction, (the reader will have no difficulty in drawing a figure,) E being the refracting edge; let i be the refracting angle; OA, OB, OC the directions of the axes of elasticity, O being any point within the prism, the two former of these lines

being in, and the latter perpendicular to, the plane of refraction; a, b, c the optical constants referring to them, that is, according to Fresnel's theory, the velocities of propagation of waves in which the vibrations are parallel to the three axes respectively. Everything being symmetrical with respect to the plane of incidence, we need only consider what takes place in that plane. This plane will cut the wave surface in a circle of radius c, and an ellipse whose semiaxes are a along oB and b along oA. We have only got to consider the ellipse, since it is it that determines the direction of the extraordinary ray. The form of the crystal will very often make known the directions of the axes of elasticity. Supposing these directions known, let α, β denote the inclinations of oA, oB to the produced parts of EA, EB respectively; α, β and i being of course connected by the equation $\alpha + \beta = \frac{\pi}{2} + i$.

Let ϕ, ψ be the angles of incidence and emergence, the light being supposed incident on the face EA; ϕ' the inclination of the refracted wave to EA, ψ' its inclination to EB, D the deviation, v the velocity of the wave within the crystal, u its velocity in the outer medium, which may be supposed to be either air, or a liquid of known refractive power. Then we have

$$D = \phi + \psi - i\,^{*} \ldots\ldots\ldots\ldots (1),$$

$$\phi' + \psi' = i \ldots\ldots\ldots\ldots (2),$$

$$v \sin\phi = u \sin\phi' \ldots\ldots\ldots\ldots (3),$$

$$v \sin\psi = u \sin\psi' \ldots\ldots\ldots\ldots (4),$$

$$v^2 = a^2 \cos^2(\alpha - \phi') + b^2 \sin^2(\alpha - \phi') \ldots\ldots (5).$$

From (2), (3), (4),

$$u \sin\psi' = v \sin\psi = u \sin(i - \phi') = u \sin i \cos\phi' - v \cos i \sin\phi;$$

$$\therefore \quad \cos\phi' = \frac{v}{u \sin i}(\sin\psi + \cos i \sin\phi);$$

and $$\sin\phi' = \frac{v}{u \sin i} \sin i \sin\phi:$$

substituting in (5),

$$u^2 \sin^2 i = a^2 \{\cos\alpha(\sin\psi + \cos i \sin\phi) + \sin\alpha \sin i \sin\phi\}^2$$
$$+ b^2 \{\sin\alpha(\sin\psi + \cos i \sin\phi) - \cos\alpha \sin i \sin\phi\}^2,$$

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* I am indebted to the Rev. P. Frost for the suggestion of employing equations (1) .... (4), rather than making use of the ellipse in which the wave surface is cut by the plane of incidence.

or $u^2 \sin^2 i = a^2 (\cos \alpha \sin \psi + \sin \beta \sin \phi)^2$
$+ b^2 (\sin \alpha \sin \psi + \cos \beta \sin \phi)^2 \ldots .(6),$
the relation between ϕ and ψ. Putting $\psi - \phi = \theta$, and taking account of (1), (6) becomes

$$2u^2 \sin^2 i = \{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha\} \{1 - \cos(D + i + \theta)\}$$
$$+ \{a^2 \sin^2 \beta + b^2 \cos^2 \beta\} \{1 - \cos(D + i - \theta)\}$$
$$+ 2(a^2 \cos \alpha \sin \beta + b^2 \sin \alpha \cos \beta) \{\cos \theta - \cos(D + i)\},$$

or $$F \cos \theta + G \sin \theta + H = 0 \ldots\ldots\ldots .(7),$$
where

$$F = a^2 \{(\cos^2 \alpha + \sin^2 \beta) \cos(D + i) - 2 \cos \alpha \sin \beta\}$$
$$+ b^2 \{(\sin^2 \alpha + \cos^2 \beta) \cos(D + i) - 2 \sin \alpha \cos \beta\},$$
$$G = (a^2 - b^2)(\sin^2 \beta - \cos^2 \alpha) \sin(D + i),$$
$$H = 2u^2 \sin^2 i - a^2 (\cos^2 \alpha + \sin^2 \beta - 2 \cos \alpha \sin \beta \cos(D + i)\}$$
$$- b^2 \{\sin^2 \alpha + \cos^2 \beta - 2 \sin \alpha \cos \beta \cos(D + i)\}.$$

Now when D, regarded as a function of θ, is a maximum or minimum $\dfrac{dD}{d\theta} = 0$, whence from (7)

$$-F \sin \theta + G \cos \theta = 0;$$

and eliminating θ from this equation and (7), we have

$$F^2 + G^2 = H^2.$$

Putting for F, G and H their values, and reducing, this equation becomes

$$\sin^2(D + i)\, a^2b^2 - \{\cos^2 \alpha + \sin^2 \beta - 2\cos(D + i)\cos \alpha \sin \beta\}\, u^2a^2$$
$$-\{\sin^2 \alpha + \cos^2 \beta - 2\cos(D+i)\sin \alpha \cos \beta\}\, u^2b^2 + \sin^2 i.u^4 = 0 \,..(8).$$

This equation will be rendered more convenient for numerical calculation by replacing products and powers of sines and cosines by sums and differences. Treated in this manner, the equation becomes

$$\text{versin } 2(D + i)\, a^2b^2 - (A + B)\, u^2a^2 - (A - B)\, u^2b^2$$
$$+ \text{versin } 2i.u^4 = 0 \ldots .(9),$$

where $A = \text{versin } D + \text{versin }(D + 2i)$,
$B = \cos 2\alpha - \cos 2\beta - \cos(D + 2\alpha) + \cos(D + 2\beta)$.

If the principal plane AOC of the crystal bisects the angle between the refracting faces, we have $\alpha = \dfrac{i}{2}$, $\beta = \dfrac{\pi}{2} + \dfrac{i}{2}$, whence from (8), putting $D + i = \Delta$,

$$\left(a^2 \sin^2 \frac{\Delta}{2} - u^2 \sin^2 \frac{i}{2}\right)\left(b^2 \cos^2 \frac{\Delta}{2} - u^2 \cos^2 \frac{i}{2}\right) = 0.$$

The former of these factors is evidently that which corresponds to the problem; the latter corresponds to refraction through a prism having its faces parallel to those of the actual prism, and having its refracting angle supplemental to i. We have therefore

$$a = u \frac{\sin \frac{i}{2}}{\sin \frac{\Delta}{2}};$$

so that the constant a is given by the same formula that applies to ordinary media, as it should.

If the refracting faces are perpendicular to the axes of elasticity which lie in the plane of incidence, the formula (8) or (9) takes a very simple form. In this case we have $\alpha = \beta = i = \frac{\pi}{2}$, and therefore

$$\cos^2 D . a^2 b^2 - u^2 a^2 - u^2 b^2 + u^4 = 0.$$

Mathematically speaking, one prism would be sufficient for determining the three constants a, b, c. For c would be determined by means of the ordinary pencil; and by observing the extraordinary pencil with the crystal in air, and again with the crystal in some liquid, we should have two equations of the form (8), by combining which we should obtain a^2 and b^2 by the solution of a quadratic equation. But since a is usually nearly equal to b, it is evident that the course of the extraordinary ray within the crystal would be nearly the same in the two observations, being in each case inclined at nearly equal angles to the refracting faces, and consequently the errors of observation would be greatly multiplied in the result. Even if a differed greatly from b, only one of these constants could be accurately determined in this manner if the refracting angle were nearly bisected by a principal plane. But two prisms properly chosen appear amply sufficient for determining accurately the three constants by the method of minimum deviations, even should neither prism have its angle exactly bisected by a principal plane of the crystal.

It is not necessary to observe the deviation when it is a minimum, as Professor Miller has remarked to me, since the angle of incidence may be measured very accurately by moving the telescope employed till the luminous slit, seen directly, appears on the cross wires, and then turning it till the slit, seen by reflection at the first face of the prism, again appears on the cross wires, the prism meanwhile remaining fixed.

The angle through which the telescope has been turned is evidently the supplement of twice the angle of incidence. If this method of observation be adopted, ϕ, D, and i will be known by observation, whence ψ will be got immediately from (1). Thus all the coefficients in (6) will be known quantities, and this equation furnishes a very simple relation between a and b. The coefficients may easily be calculated numerically by treating them like those in equation (8), or else by employing subsidiary angles.

SUR LA REPRÉSENTATION GÉOMÉTRIQUE DES FONCTIONS ELLIPTIQUES DE PREMIÈRE ESPÈCE.

Par J. ALFRED SERRET, (de Paris).

§. I. ON sait que la fonction Elliptique de première espèce au module $\sqrt{\frac{1}{2}}$ est *identique* à l' arc de Lemniscate considéré comme fonction du rayon vecteur central, ou d' une fonction convenablement déterminée de ce rayon vecteur; mais malgré les travaux de Legendre cette propriété n' avait pu être généralisée jusqu' ici, et l' en ne connaissait aucune autre courbe algébrique dont l' arc considéré comme fonction d' une coordonée convenable fût identiquement representé par une fonction elliptique de première espèce.

J'ai résolu le premier ce problème, dans plusieurs mémoires presentés à l' Academie des Sciences et qui ont été publiés depuis dans le *Journal des Mathématiques* de M. Liouville. Voici en peu de mots quel a été l' esprit de mes nouvelles recherches.

Les coordonnées rectangulaires de la Lemniscate sont exprimables rationnellement en fonction de l' amplitude de la fonction elliptique qui représente l' arc; si en effet l' on pose

$$x = a\sqrt{2}\,\frac{z + z^3}{1 + z^4}, \quad y = a\sqrt{2}\,\frac{z - z^3}{1 + z^4};$$

on aura
$$\sqrt{(dx^2 + dy^2)} = 2a\,\frac{dz}{\sqrt{(1 + z^4)}},$$

et il est bien facile de s' assurer que les équations précédentes sont relatives à la Lemniscate: cette remarque m' a conduit naturellement a chercher les solutions *réelles et rationnelles* que peut admettre une équation de la forme

$$dx^2 + dy^2 = Zdz^2;$$

La discussion de cette équation se trouve dans mon premier mémoire, mais je me bornerai à rappeler les résultats auxquels je suis parvenu relativement à la représentation géométrique des fonctions elliptiques de première espèce: dans ce cas, je fais $z = \dfrac{C^2}{(z^2 - a^2)(z^2 - \alpha^2)}$, C étant une constante reelle, a et α deux constantes essentiellement imaginaires et conjuguées, en sorte que l' equation à resoudre devient

$$dx^2 + dy^2 = C^2 \frac{dz^2}{(z^2 - a^2)(z^2 - \alpha^2)} \ldots\ldots\ldots (1).$$

Il est visible que l' on satisfera à cette équation differentielle indeterminée, en posant

$$\left.\begin{aligned} x + y\sqrt{(-1)} &= Ce_{\omega}^{\sqrt{-1}} \int \frac{(z-a)^m (z+a)^n}{(z-\alpha)^{m+1}(z+\alpha)^{n+1}}\, dz, \\ x - y\sqrt{(-1)} &= Ce^{-\omega\sqrt{(-1)}} \int \frac{(z-\alpha)^m (z+\alpha)^n}{(z-a)^{m+1}(z+a)^{n+1}}\, dz \end{aligned}\right\} \ldots\ldots (2),$$

ou m et n sont des nombres entiers et positifs, et ω un angle réel quelconque. Pour que les intégrales précédentes soient algébriques, *il faut et il suffit* que la quantité

$$\zeta = \frac{(a + \alpha)^2}{4a\alpha},$$

soit une racine de l' equation

$$\frac{d^n \zeta^m (1 - \zeta)^n}{d\zeta^n} = 0 \ldots\ldots\ldots\ldots\ldots (3),$$

en supposant, ce qui est permis, que m ne soit pas supérieur à n. L' équation (3) a toutes ses racines réelles positives et inférieures à 1, ce qui est un théorème d' une bien grande importance, car autrement les quantités a et α ne seraient plus conjugueés, et les équations (2) ne se changeraient plus l'une en l'autre par le changement de $\sqrt{(-1)}$ en $-\sqrt{(-1)}$; au surplus il est facile de démontrer à priori que l'equation (1) n' admet aucune solution rationnelle et réelle si a et α sont réels.

n restant indeterminé, si l'on donne successivement à m les valeurs

$$1, 2, 3, \ldots\ldots$$

on aura une multitude de classes de courbes renfermant chacune une infinité de courbes individuelles, dont l' arc est identiquement représenté par une fonction elliptique de première espèce; les modules des fonctions dont il s' agit ont pour carrés les racines ζ de l'équation (3).

Je me suis plus spécialement occupé des courbes de la première classe pour les quelles $m = 1$; on a dans ce cas

$$x + y\sqrt{(-1)} = Ce^{\omega\sqrt{(-1)}}\,\frac{(z+a)^{n+1}}{(z-\alpha)(z+\alpha)^n},$$

$$x - y\sqrt{(-1)} = Ce^{-\omega\sqrt{(-1)}}\,\frac{(z+\alpha)^{n+1}}{(z-a)(z+a)^n},$$

d' où
$$x^2 + y^2 = C^2\,\frac{(z+a)(z+\alpha)}{(z-a)(z-\alpha)};$$

avec la condition
$$\frac{(a+\alpha)^2}{4a\alpha} = \frac{n}{n+1},$$

et l'en en déduit par la différentiation

$$dx + \sqrt{(-1)}\,dy = Ce^{\omega\sqrt{(-1)}}\{-(a+\alpha) - n(a-\alpha)\}\,\frac{(z-a)(z+a)^n}{(z-\alpha)^2(z+\alpha)^{n+1}}\,dz,$$

$$dx - \sqrt{(-1)}\,dy = Ce^{-\omega\sqrt{(-1)}}\{-(a+\alpha) + n(a-\alpha)\}\,\frac{(z-\alpha)(z+\alpha)^n}{(z-a)^2(z+a)^{n+1}}\,dz,$$

d'ou
$$ds = 2C\sqrt{(na\alpha)}\,\frac{dz}{\sqrt{\{(z+a)(z+\alpha)(z-a)(z-\alpha)}}.$$

Si r désigne le rayon vecteur $\sqrt{(x^2+y^2)}$, et si l'on suppose pour simplifier qu' en prenne pour unité le paramètre C, on aura

$$ds = 2\sqrt{\{n(n+1)}\,\frac{dr}{\sqrt{\{-r^4 + 2(2n+1)r^2 - 1\}}} \quad \ldots. \ (4),$$

ce qui donne ce theorème remarquable : l'arc des courbes de première classe est une fonction elliptique du rayon vecteur. L'équation précédente n' est autre chose que l'équation différentielle des courbes de première classe, lesquelles ont à leur tête la Lemniscate qui correspond au cas de $n = 1$.

Mais ici se présente une remarque bien importante due à M. Liouville, et qui consiste en ce que le nombre n, que la nature de notre analyse obligeait à être entier, n'a réellement besoin que d' etre commensurable, afin que nos courbes ne cessent pas d' être algébriques, et cette observation s' applique egalement aux classes suivantes, en sorte qu' on pourra à l' aide des courbes de la première classe seulement, representer toutes les fonctions elliptiques dont les modules ont pour carrés des nombres rationnels quelconques ; au surplus on verra plus loin que les courbes de l' equation (4) sont soumises à un mode uniforme de génération, que n soit entier ou fractionnaire, ou même incommensurable.

Si θ designe la seconde coordonné polaire, l'equation (4) donnera

$$d\theta = \frac{r^2 - (2n+1)}{\sqrt{\{-r^4 + 2(2n+1)r^2 - 1\}}} \frac{dr}{r} \ldots\ldots (5),$$

d'ou l'on déduit

$$\int \tfrac{1}{2} r^2 \, d\theta = -\tfrac{1}{4}\sqrt{\{-r^4 + 2(2n+1)r^2 - 1\}} + \text{constante}$$

équation qui montre que les courbes de la première classe sont toutes carrables.

Ou trouve pour l'intégrale de l'équation (5)

$$\theta = \theta_0 - \lambda - (2n+1)\mu,$$

θ_0 désignant une constante arbitraire, λ et μ deux angles compris entre 0 et $\frac{\pi}{2}$ et déterminés par les équations

$$r^2 = (2n+1) + 2\sqrt{\{n(n+1)\}} \cos 2\lambda,$$

$$\frac{1}{r^2} = (2n+1) + 2\sqrt{\{n(n+1)\}} \cos 2\mu,$$

et l'en conclut aisément de là, l'équation de nos courbes en coordonnées polaires.

§ II. Je viens d'analyser d'une manière succincte les différens résultats que j'ai publiés dans mes derniers mémoires; depuis, une étude plus approfondie des formules dont je n'ai pu indiquer ici qu'une partie, m'a conduit à deux propriétés géométriques remarquables communes à toutes les courbes elliptiques de la première classe, et qui fournissent pour ces courbes un mode uniforme de génération d'une extrême élégance; mais, bien que j'aie découvert ces propriétés, comme je viens de le dire, en suivant le cours naturel de mes recherches analytiques, je préfère, me placer ici à un point de vue différent, afin que les résultats qui vont suivre deviennent entièrement indépendants de ces considérations analytiques, et soient en conséquence parfaitement compris des lecteurs qui n'auraient pas connaissance de mes recherches antérieures.

Au nouveau point de vue ou je me place, je commence par démontrer ces propriétés sur la Lemniscate, et je les généralise ensuite bien aisément.

Théorème I. Soit r le rayon vecteur issu de l'un des foyers d'une Lemniscate, dont la demie distance focale est prise pour unité, et dont le demi axe sera dèslors $\sqrt{2}$; on pourra toujours construire un triangle dont les cotés seront respectivement r, 1 et $\sqrt{2}$, car le rayon vecteur reste compris entre $\sqrt{2} - 1$ et $\sqrt{2} + 1$: cela posé, si a designe l'angle de ce

triangle opposé au coté $\sqrt{2}$, et β celui qui est opposé au coté 1, l'angle polaire θ que forme le rayon vecteur de la Lemniscate avec l'axe, sera toujours donné par l'équation

$$\cos\theta = \cos(\alpha - 2\beta),$$

Remarque. Soit O l'origine, c'est à dire l'un des foyers de la Lemniscate, et OM un rayon vecteur quelconque; construissons le triangle OMP, de telle sorte que

$$OP = 1 \text{ et } MP = \sqrt{2},$$

(ce triangle peut être fait d'un coté ou de l'autre de OM, cela importe peu en ce moment), puis imaginons que le point M décrive d'un mouvement continu la Lemniscate entière, le point P qu'on peut toujours supposer se mouvoir d'un mouvement continu, décrira deux fois la circonférence tracée de l'origine comme centre avec l'unité comme rayon.

Corollaire. Du théorème précédent qu'on démontre bien aisément, on déduit la génération suivante de la Lemniscate.

Soit OMP *un triangle dont le sommet* O *est fixe dont les cotés mobiles* OP *et* MP *sont constamment égaux, l'un à* 1, *l'autre à* $\sqrt{2}$; *si l'on fait varier ce triangle de telle sorte que le cosinus de l'angle que fait le coté variable* OM *avec une droite fixe soit constamment égal au cosinus de l'angle* MOP − 2OMP, *le point* M *engendrera une Lemniscate dont* O *sera un foyer, et la droite fixe l'axe.*

Théorème II. Soit comme précédemment OM un rayon vecteur de la Lemniscate, et construisons le triangle OMP de part et d'autre du rayon OM, la tangente en M à la Lemniscate passera constamment par le centre du cercle circonscrit à l'un de ces triangles; si en outre on considère spécialement celui de ces triangles pour le quel cette propriété a lieu, et qu'en vertu du théorème I. on le fasse servir à la description de la Lemniscate par un *mouvement continu*, cette propriété se conservera pour tous les points de la courbe.

Remarque. Ce Théorème donne un moyen très simple de construire la tangente en un point de la courbe, car il suffira de construire le triangle correspondant à ce point, et de le joindre au centre du cercle circonscrit au triangle, mais il conduit aussi à un nouveau mode de génération pour la Lemniscate.

Théorème III. Soit OMP un triangle dont le sommet O est fixe, et dont les cotés OP et MP sont constamment égaux, l'un à 1 l'autre à $\sqrt{2}$, le sommet M décrira une lemniscate, si son deplacement infiniment petit MM' a constamment lieu suivant le rayon CM du cercle circonscrit au triangle OMP.

Remarque. Le triangle dont nous venons de parler, joue, comme on voit, un rôle assez important dans la théorie de la lemniscate, aussi je ne crois pas inutile de mentionner une dernière propriété, qui consiste en ce que l'aire de ce triangle et l'aire du secteur de la courbe ont la même différentielle.

§. III. Soit maintenant n un nombre entier ou fractionnaire, ou même incommensurable et construisons le triangle OMP tel que

$$OP = \sqrt{n} \quad \text{et} \quad MP = \sqrt{(n+1)}.$$

Puis imaginons que le sommet O restant fixe, le triangle varie de telle sorte que le cosinus de l'angle θ formé par le seul coté variable OM avec une droite fixe, soit constamment égal au cosinus de l'angle

$$n.MOP - (n+1)\ OMP,$$

le point M engendrera une courbe (algébrique si n est commensurable) dont l'arc sera une fonction elliptique du rayon vecteur, réductible au module $\sqrt{\left(\frac{n}{n+1}\right)}$,

Soit en effet $MOP = \alpha$, $OMP = \beta$, l'équation de la courbe résultera de l'éliminations de α et β entre

$$\cos\theta = \cos\{n\alpha - (n+1)\beta\}$$

$$\left\{\begin{aligned} \cos\alpha &= \frac{r^2-1}{2r\sqrt{n}}, \\ \cos\beta &= \frac{r^2+1}{2r\sqrt{(n+1)}}, \end{aligned}\right. \quad \text{d'où} \quad \left\{\begin{aligned} \sin\alpha &= \frac{\Delta}{2r\sqrt{n}}, \\ \sin\beta &= \frac{\Delta}{2r\sqrt{(n+1)}}, \end{aligned}\right.$$

en faisant pour abréger

$$\Delta = \sqrt{\{-r^4 + 2(2n+1)r^2 - 1\}}.$$

cela posé on déduit par la différentiation

$$\pm d\theta = n d\alpha - (n+1)\, d\beta,$$

et $$d\alpha = -\frac{r^2+1}{\Delta}\frac{dr}{r}, \quad d\beta = -\frac{r^2-1}{\Delta}\frac{dr}{r},$$

d'où $$d\theta = \frac{r^2-(2n+1)}{\Delta}\frac{dr}{r},$$

et par suite $$ds = 2\sqrt{\{n(n+1)}\frac{dr}{\Delta}.$$

Des équations précédentes on déduit encore les formules suivantes qu'il convient de remarquer;

$$ds = -\sqrt{n}\frac{d\alpha}{\cos\beta}, \quad ds = -\sqrt{(n+1)}\frac{d\beta}{\cos\alpha}:$$

on a d'ailleurs en posant $k = \sqrt{\left(\frac{n}{n+1}\right)}$,

$$\sin\beta = K\sin\alpha, \quad \text{d'ou} \quad \cos\beta = \sqrt{(1 - K^2\sin^2\alpha)},$$

donc
$$ds = -\sqrt{n}\,\frac{d\alpha}{\sqrt{(1 - K^2\sin^2\alpha)}},$$

et l'arc de courbe compté à partir du point de l'axe polaire qui correspond à $\alpha = 0$ ou $r = \sqrt{(n+1)} \pm \sqrt{n}$, sera exprimé par l'intégral elliptique au module K

$$\sqrt{n}\int_0^{\alpha}\frac{d\alpha}{\sqrt{(1 - K^2\sin^2\alpha)}},$$

ce qu'il s'agissait de démontrer.

On voit aisément que dans le cas de $n = 1$, la courbe dont nous parlons se confond avec la lemniscate de Bernouilli et l'on a ainsi la démonstration du théorème I, du paragraphe II.

Il est bon de remarquer encore que l'aire du triangle générateur OMP est $\frac{\Delta}{4}$, et l'on trouve aisément

$$\int\tfrac{1}{2}r^2 d\theta = \frac{\Delta}{4} + \text{constant},$$

d'où l'on conclut que l'aire du secteur du courbe compté à passer de l'axe polaire est toujours égale à l'aire du triangle générateur.

Je passe maintenant à l'examen de la seconde propriété de ces courbes remarquables : on a dans le triangle OMP

$$r^2 = 2n + 1 + 2\sqrt{\{n(n+1)\}}\cos(\alpha + \beta),$$

d'où
$$\cos(\alpha + \beta) = \frac{r^2 - (2n+1)}{2\sqrt{\{n(n+1)\}}} = \frac{rd\theta}{ds},$$

et
$$\sin(\alpha + \beta) = \frac{\Delta}{2\sqrt{\{n(n+1)\}}} = \frac{dr}{ds},$$

d'où l'on conclut que l'inclinaison de la normale sur le rayon vecteur est précisément egale à $\alpha + \beta$, si donc on fait au point M un angle $PMN = MOP$, MN sera la normale au point M de la courbe lequel correspond à la position OMP du triangle générateur. D'ailleurs le point O se trouve nécessairement sur le segment capable de l'angle PMN, que l'on décrivait sur MP, ce qui montre que MN est tangente au cercle circonscrit au triangle générateur, et si C est le centre du cercle circonscrit, le rayon MC sera précisément la tangente à la courbe.

De ce qui precede résulte le mode de génération suivant pour les courbes elliptiques :—*Si le triangle OMP varie de telle manière que le sommet O reste fixe, que les deux cotés OP et MP soient constamment égaux le premier à* $\sqrt{n}$, *le second à* $\sqrt{(n+1)}$, *et que de plus le déplacement, infiniment petit MM' du point M ait lieu à chaque instant suivant la droite qui joint ce point au centre du cercle circonscrit, au triangle générateur, le point M engendrera la courbe elliptique que correspond au nombre n.*

On a ainsi en particulier la démonstration des théorèmes II. et III. du paragraphe II.; lesquels sont relatifs seulement à la lemniscate.

On obtient aisément l' expression du rayon de courbure; soit ε l'angle que fait la normal avec l' axe polaire, on aura

$$\varepsilon = \theta - (\alpha + \beta)$$

car $(\alpha + \beta)$ est l'angle de la normal avec le rayon vecteur; on a, en différentiant, l' angle de contingence $d\varepsilon$

$$d\varepsilon = d\theta - d\alpha - d\beta = \frac{3r^2 - (2n+1)}{\Delta}\,\frac{dr}{r},$$

et pour le rayon de courbure

$$\frac{ds}{d\varepsilon} = R = \frac{2r\sqrt{\{n(n+1)}}{3r^2 - (2n+1)},$$

§ IV. Les courbes dont je viens de parler sont celles que j'ai désignées sous le nom de courbes elliptiques de la première classe, dans une note insérée au Journal de M. Liouville (t. x. 1846); on voit qu'il s'en trouve une dont l'arc sera identique à telle fonction elliptique de première espèce, que l'on voudra. Les courbes de la troisième classe sont définies par l' équation

$$x + y\sqrt{(-1)} = ce^{\omega\sqrt{-1}} \int \frac{(z-a)^m (z+a)^n}{(z-\alpha)^{m+1}(z+\alpha)^{n+1}}\,dz$$

où la quantité $\dfrac{(a+\alpha)^2}{4a\alpha} = \zeta$, est une racine de l'équation

$$\frac{d^n \zeta^m (1-\zeta)^n}{d\zeta^n} = 0.$$

Si l' on a $n = m =$ un nombre entier impair $2\mu + 1$, cette équation aura toujours pour racine $\frac{1}{2}$, en sorte que toutes les classes de rang impair comprendront une courbe dont l' arc sera identique à l'arc de lemniscate. On a dans ce cas $a^2 = -\alpha^2 = \sqrt{(-1)}$, et ces nouvelles courbes sont définies par l'équation

$$x + y\sqrt{(-1)} = ce^{\omega\sqrt{(-1)}} \int \frac{\{z^2 - \sqrt{(-1)}\}^{2\mu+1}}{(z^2 + \sqrt{-1})^{2\mu+2}}\,dz,$$

Il est facile de voir *a posteriori* que l'intégrale du second membre est algébrique, car en posant

$$\frac{2z\sqrt[4]{(-1)}}{z^2+\sqrt{(-1)}}=t$$

on trouve $\quad x+y\sqrt{(-1)}=-\dfrac{ce^{\omega i}}{\sqrt[4]{(-1)}}\int(1-t^2)^\mu\,dt.$

Quant à l'arc de cette courbe, on aura évidemment pour sa différentielle $\quad \sqrt{(dx^2+dy^2)}=c\dfrac{dz}{\sqrt{(1+z^4)}},$

quel que soit le nombre entier μ.

Paris, le 28 *Mars*, 1846.

ON THE PRINCIPAL AXES OF A SOLID BODY.

By WILLIAM THOMSON.

(*Continued from* § 12, p. 133.)

§§ 13, 16. Second method of conducting part of the preceding investigation. § 14. Conditions for the equality of two of the principal moments of inertia. § 15. Remarks on the conditions for the existence of principal axes in more than one of the coordinate planes. § 17. Comparison of the different formulæ. § 18. Formulæ for the determination of the principal axes and principal moments of inertia relative to different points of a body. § 19. Geometrical representation of the preceding results by means of confocal surfaces of the second order. §§ 20...22. Equimomental surface and curves.

IN the second method alluded to, either of the forms of the equations of condition for principal axes given above may be made use of; but, on account of the application which is to be made, it will be more convenient, in what follows, to employ the first form (equations (8), § 6).

13. Let

$$\left.\begin{array}{lll} A'=gh, & B'=hf, & C'=fg \\ F=f^2+\alpha, & G=g^2+\beta, & H=h^2+\gamma \end{array}\right\}\ldots\ldots(a),$$

from which we deduce

$$\left.\begin{array}{lll} f^2=\dfrac{B'C'}{A'}, & g^2=\dfrac{C'A'}{B'}, & h^2=\dfrac{A'B'}{C'} \\ \alpha=F-\dfrac{B'C'}{A'}, & \beta=G-\dfrac{C'A'}{B'}, & \gamma=H-\dfrac{A'B'}{C'} \end{array}\right\}\ldots\ldots(b).$$

These latter equations (since A', B', C' are positive) shew that the values of the constants f, g, h, α, β, γ, deduced from

the given constants, are necessarily real. By means of these assumptions, and by dividing the first of equations (8) by mn, the second by nl, the third by lm, we reduce them to the two following:

$$\frac{f(fl+gm+hn)+\alpha l}{l}=\frac{g(fl+gm+hn)+\beta m}{m}$$
$$=\frac{h(fl+gm+hn)+\gamma n}{n}\quad\ldots.(15).$$

If we put $$S=fl+gm+hn\ \ldots\ldots\ldots\ldots\ldots\ (16),$$
and denote each member of equations (15) by K, we find

$$l=\frac{Sf}{K-\alpha},\quad m=\frac{Sg}{K-\beta},\quad n=\frac{Sh}{K-\gamma}\ \ldots.(17);$$

and therefore (except in the case of $S=0$, when some of the principal axes are indeterminate), we have, by (16),

$$\frac{f^2}{K-\alpha}+\frac{g^2}{K-\beta}+\frac{h^2}{K-\gamma}=1\ \ldots\ldots.(18).$$

This equation determines three real values for K, (see First Series, vol. IV. p. 229), one of which lies between γ and β, another between β and α, and the third between α and ∞* (α, β, γ being supposed to be in order of descending magnitude). Any one of these values, substituted for K in (15) give values of the ratios $l:m:n$, which fix the position of a principal axis. Hence there are three, and only three, principal axes through any point. These may be shewn to form a rectangular system, in the following manner, which is quite similar to a method given in the *Mathematical Journal* (First Series, vol. III. p. 291), for demonstrating the perpendicularity of two lines in space, in a corresponding case.

Let K_1, K_2 be two roots of the cubic (18), and (l_1, m_1, n_1) (l_2, m_2, n_2) the corresponding principal axes. Substituting for K the values K_1 and K_2 successively, in (18), we obtain, by subtraction,

$$(K_1-K_2)\left\{\frac{f^2}{(K_1-\alpha)(K_2-\alpha)}+\frac{g^2}{(K_1-\beta)(K_2-\beta)}+\frac{h^2}{(K_1-\gamma)(K_2-\gamma)}\right\}=0.$$

Unless K_1 be equal to K_2 the second factor must vanish, and therefore, by (17),

$$l_1l_2+m_1m_2+n_1n_2=0;$$

which shews that the principal axes corresponding to any two different roots of the cubic are at right angles. Hence if the

* If one of the quantities f, g, or h vanishes, there will be a root equal to the corresponding quantity α, β, or γ.

cubic has three unequal roots, the three principal axes form a rectangular system.

14. If two roots of the cubic equation be equal, only one principal axis will be determinate, and every line through the origin, perpendicular to it, will, as may be easily shewn, be also a principal axis, and in this case the body will, with respect to dynamics, be in the same condition as if it were symmetrical round an axis. Now if two roots be equal, each must be equal to one of the quantities α, β, γ, on account of the limits stated above. Hence, and by clearing the equation of fractions, we find that if two roots are equal the conditions

$$\alpha = \beta = \gamma ;$$

or, in terms of the original coefficients,

$$F - \frac{B'C'}{A'} = G - \frac{C'A'}{B'} = H - \frac{A'B'}{C'}$$

must in general be satisfied.* The same conditions may be found by equating to zero the expression, in terms of the coefficients, for the product of the squares of the differences of the roots of the cubic equation, which has been given in a very remarkable form, as the sum of seven squares, by Kummer (*Crelle's Journal*, vol. XXI. p. 74). If we employ the quantities $\alpha, \beta, \gamma, f, g, h$, instead of the six coefficients F, G, H, A', B', C', and denote, by K_1, K_2, K_3, the roots of the cubic, the result which he has obtained may be written as follows:†

$$\begin{aligned}
&(K_2 - K_3)^2 (K_3 - K_1)^2 (K_1 - K_2)^2 \\
&\qquad = 15 f^2 g^2 h^2 \{f^2 (\beta - \gamma)^2 + g^2 (\gamma - \alpha)^2 + h^2 (\alpha - \beta)^2\} \\
&+ g^2 h^2 \{f^2 (2\alpha - \beta - \gamma) + 2g^2 (\gamma - \alpha) + 2h^2 (\beta - \alpha) - 2(\gamma - \alpha)(\beta - \alpha)\}^2 \\
&+ h^2 f^2 \{g^2 (2\beta - \gamma - \alpha) + 2h^2 (\alpha - \beta) + 2f^2 (\gamma - \beta) - 2(\alpha - \beta)(\gamma - \beta)\}^2 \\
&+ f^2 g^2 \{h^2 (2\gamma - \alpha - \beta) + 2f^2 (\beta - \gamma) + 2g^2 (\alpha - \gamma) - 2(\alpha - \gamma)(\beta - \alpha)\}^2 \\
&+ \{[f^2 (f^2 + 2\alpha - \beta - \gamma) + g^2 h^2] (\beta - \gamma) \\
&+ [g^2 (g^2 + 2\beta - \gamma - \alpha) + h^2 f^2] (\gamma - \alpha) + [h^2 (h^2 + 2\gamma - \alpha - \beta) + f^2 g^2] (\alpha - \beta)\}^2 .
\end{aligned}$$

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* If however (two of the quantities A', B', C' beingequal to nothing) f, g, or h vanishes, the equation will have equal roots if a second root be equal to the root α, β, or γ implied by this circumstance. Thus if $B' = 0$ and C', which give $f = 0$, we have, as the additional condition for equal roots,

$$\frac{g^2}{\alpha - \beta} + \frac{h^2}{\alpha - \gamma} = 1,$$

or, which is equivalent, $A'^2 = (F - G)(F - H)$.

† It has been shewn by Borchhardt (*Crelle*, XXX. p. 38) that Kummer's result is the particular case of a property of the equation (of the n^{th} degree) which occurs in the reduction of homogeneous functions of the second order (of n variables).

Since all the quantities $\alpha, \beta, \gamma, f, g, h$ are, in the actual problem, whether of the reduction of the general equation of the second degree, or of the determination of principal axes of a solid, necessarily real, each of the seven squares must vanish, if the sum vanishes. Hence we obtain seven equations as the conditions for the equality of two of the roots of the cubic, which, the special cases of any of the quantities f, g, h vanishing being excluded, are equivalent to the two distinct equations

$$\alpha = \beta = \gamma.$$

If imaginary values of the coefficients were admissible, one condition would of course be sufficient.

15. In the first part of this paper (§ 10) it was shewn that the condition for there being a principal axis in the plane of (xy) is

$$B'(FA' - B'C') - A'(GB' - C'A') = 0,$$

(which is the same as (a) § 10, since $F - G = B - A$). Each member of this equation may be divided by $A'B'$, unless either A' or B' vanishes, in which case, to satisfy the equation, another also of the three quantities A', B', C' must vanish, and one of the axes of coordinates is a principal axis. Hence, if none of the axes of coordinates be a principal axis, the condition that there may be a principal axis in the plane of (xy) is

$$F - \frac{B'C'}{A'} = G - \frac{C'A'}{B'} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (a).$$

Similarly the condition that there may be a principal axis in the plane of yz, is

$$G - \frac{C'A'}{B'} = H - \frac{A'B'}{C'} \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (b).$$

If there be a principal axis in each of the planes (xy) and (yz), equations (a) and (b) will hold simultaneously, and therefore the conditions are

$$F - \frac{B'C'}{A'} = G - \frac{C'A'}{B'} = H - \frac{A'B'}{C'} \;.\;.\;.\;.\;.\;.(c).$$

From the symmetry of the three members of these equations we infer that, if there be a principal axis in the plane (xy), and another in (yz), there shall also be a principal axis in the plane of (zx), and the conditions for this case are the same as those for two of the roots of the cubic being equal.

These results might also have been arrived at by the following simple considerations. It is impossible to find, in the planes of (xy) and (yz), two lines at right angles, of which neither is one of the axes of coordinates. Hence,

none of the axes of coordinates being principal axes, if there be a principal axis in xy, and another in (yz), these two cannot be at right angles, and therefore the roots of the cubic from which they are deduced must be equal. Also every line through the origin, in their plane, is a principal axis, and therefore there is a principal axis in the plane of (zx), (where it is cut by this plane of principal axes).

16. The roots K_1, K_2, K_3 of the cubic equation (18) are not the moments of inertia round the principal axes, but are quantities relative to these axes which correspond to F, G, H, round the axes of coordinates. For, if we take $x'=lx+my+nz$, we have

$$\Sigma\mu x'^2 = l^2\Sigma\mu x^2 + m^2\Sigma\mu y^2 + n^2\Sigma\mu z^2 + 2(mn\Sigma\mu yz + nl\Sigma\mu zx + lm\Sigma\mu xy)$$
$$= Fl^2 + Gm^2 + Hn^2 + 2(A'mn + B'nl + C'lm),$$

which is the value of each member of equations (15). Hence the moment of inertia round any one of the principal axes is the sum of the roots of the cubic corresponding to the other two. But the sum of the roots of the cubic (18) is $f^2 + g^2 + h^2 + \alpha + \beta + \gamma$, or $F + G + H$: hence, if P_1 be the moment of inertia round the axis corresponding to K_1, we have

$$P_1 = K_2 + K_3$$
$$= F + G + H - K_1 = \tfrac{1}{2}(A + B + C) - K_1.$$

17. From the last section it follows that, P_1, P_2, P_3 being the three roots of the cubic (13) given in the first part of this paper, and K_1, K_2, K_3 the corresponding roots of (18), we have

$$P_1 = F + G + H - K_1, \quad P_2 = F + G + H - K_2,$$
$$P_3 = F + G + H - K_3.$$

Hence if in the equation (18) we take $K = F + G + H - P$, we obtain an equation differing only in form from (13). Again, if we had treated equations (8) in the same manner as equations (9) were treated (in § 7), we should have obtained an equation similar in form to (13), and not differing from (18) but in form. Thus we have the following four equations, by means of any one of which the principal axes and moments might be determined:

$$\left\{\begin{aligned} &(A-P)(B-P)(C-P) - A'^2(A-P) - B'^2(B-P) - C'^2(C-P) \\ &\qquad\qquad - 2A'B'C' = 0 \ldots(a) \\ &\frac{f^2}{P-\left(A+\dfrac{B'C'}{A'}\right)} + \frac{g^2}{P-\left(B+\dfrac{C'A'}{B'}\right)} + \frac{h^2}{P-\left(C+\dfrac{A'B'}{C'}\right)} + 1 = 0 \ldots(a') \end{aligned}\right.$$

$$\left\{\begin{aligned}&(F-K)(G-K)(H-K)-A'^2(F-K)-B'^2(G-K)-C'^2(H-K)\\&\qquad\qquad\qquad\qquad\qquad\qquad+2A'B'C'=0\ldots(b)^*\\&\frac{f^2}{K-\left(F-\frac{B'C'}{A'}\right)}+\frac{g^2}{K-\left(G-\frac{C'A'}{B'}\right)}+\frac{h^2}{K-\left(H-\frac{A'B'}{C'}\right)}-1=0\ldots(b'),\end{aligned}\right.$$

the quantities F, G, H, f, g, h, being given by the equations

$$F=\tfrac{1}{2}(B+C-A),\quad G=\tfrac{1}{2}(C+A-B),\quad H=\tfrac{1}{2}(A+B-C),$$

$$f^2=\frac{B'C'}{A'},\quad g^2=\frac{C'A'}{B'},\quad h^2=\frac{A'B'}{C'}.$$

It may be algebraically verified that (a) and (a') are identical, as also (b) and (b'); and that (b) or (b') may be deduced from (a) or (a') by assuming

$$P=F+G+H-K\ldots\ldots\ldots\ldots(c).$$

The roots of (a) or (a') are the three principal moments of inertia, and the roots of (b) or (b') substituted in (c), give the same quantities.

18. I shall conclude this paper by applying some of the formulæ given above to the solution of the following problem.

"Having given the moments of inertia of a body round the principal axes through its centre of gravity, shew how to determine the position of the principal axes through any other point, and the moments of inertia round them."—*St. Peter's College Examination Papers*, May 1845.

Let O be the centre of gravity; OX, OY, OZ principal axes; A, B, C the moments of inertia of the body round them: and, according to our previous notation, let

$$F=\Sigma\delta\mu x^2,\quad G=\Sigma\delta\mu y^2,\quad H=\Sigma\delta\mu z^2,$$

so that $\quad A=G+H,\quad B=H+F,\quad C=F+G\ldots.(a).$

Let P be any point (ξ, η, ζ) for which it is required to determine the principal axes and moments. The integrals (or sums) which will enter as coefficients in the equations for determining the required quantities will be

$$\Sigma\delta\mu\,(x-\xi)^2,\quad \Sigma\delta\mu\,(y-\eta)^2,\quad \Sigma\delta\mu\,(z-\zeta)^2,$$

$$\Sigma\delta\mu\,(y-\eta)(z-\zeta),\quad \Sigma\delta\mu\,(z-\zeta)(x-\xi),\quad \Sigma\delta\mu\,(x-\xi)(y-\eta).$$

Expanding these expressions, and taking into account the properties of the axes of coordinates (principal axes through the centre of gravity), we find that they are respectively equal to

$$F+\mu\xi^2,\quad G+\mu\eta^2,\quad H+\mu\zeta^2,$$

$$\mu\eta\zeta,\qquad \mu\zeta\xi,\qquad \mu\xi\eta,$$

* This is the form of the "discriminating cubic" usually given. (See First Series, vol. I. p. 35; or *Earnshaw's Dynamics*, Art. 190.)

Hence, by equations (a) of § 13, and by (13), (17), and (18), which are directly applicable to this case, after the necessary change in notation, and a convenient modification of equations (17), we obtain

$$\frac{\mu\xi^2}{K-F}+\frac{\mu\eta^2}{K-G}+\frac{\mu\zeta^2}{K-H}=1 \quad \ldots\ldots\ldots (b).$$

$$\frac{l}{\dfrac{\xi}{K-F}}=\frac{m}{\dfrac{\eta}{K-G}}=\frac{n}{\dfrac{\zeta}{K-H}} \quad \ldots\ldots\ldots\ldots (c),$$

which contain the solution of the problem, as (b) determines three real values for K, any one of which substituted in (c) gives the ratios $l:m:n$, fixing the position of a principal axis, and subtracted from the sum of the roots, leaves as remainder the moment of inertia round this axis.

19. These equations lead to a very simple geometrical construction for the principal axes through P; but before stating this, it will be convenient to make a slight modification in their form, by means of the following assumptions.

Let $\qquad \mu Q = K - (F+G+H) \quad \ldots\ldots\ldots\ldots (c'),$

and let $\qquad A = \mu a^2, \quad B = \mu b^2, \quad C = \mu c^2.$

The equations then become

$$\frac{\xi^2}{Q+a^2}+\frac{\eta^2}{Q+b^2}+\frac{\zeta^2}{\mu+c^2}=1 \quad \ldots\ldots\ldots (d),$$

$$\frac{l}{\dfrac{\xi}{Q+a^2}}=\frac{m}{\dfrac{\eta}{Q+b^2}}=\frac{n}{\dfrac{\zeta}{Q+c^2}} \quad \ldots\ldots\ldots\ldots (e).$$

If now we suppose a value of Q to be substituted in (d), which satisfies the equation when given values are assigned to ξ, η, ζ, and consider Q to remain constant, when ξ, η, ζ vary, the locus of ($\xi\,\eta\,\zeta$) will be a surface of the second order, confocal with the ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots\ldots\ldots\ldots\ldots (f),$$

and passing through the given point, and equations (e) determine the direction-cosines of a normal to this surface at the point $\xi\,\eta\,\zeta$. The three roots of equation (d), considered as a cubic for determining Q, when ξ, η, ζ have any given values, correspond to the three surfaces of the second order, the ellipsoid, the hyperboloid of one sheet, and the hyperboloid of two sheets passing through the point ($\xi\,\eta\,\zeta$), and confocal

with the surface (f), which we shall, for distinction, call the *central ellipsoid* of the body.*

The principal axes through any point of a solid body are normals to the three surfaces of the second order confocal with the central ellipsoid, which intersect in that point.†

For determining the principal moments corresponding to the point ($\xi\,\eta\,\zeta$), since $F + G + H + \mu(\xi^2 + \eta^2 + \zeta^2)$ is the sum of the roots of the cubic (b), we have

$$P_1 = F + G + H + \mu(\xi^2 + \eta^2 + \zeta^2) - K_1,$$

where P_1 is the moment of inertia round the axis given by the root K_1 of the cubic. Hence if we take

$$K = F + G + H + \mu(\xi^2 + \eta^2 + \zeta^2) - P \;\ldots\ldots (g)$$

in (d), the equation thus obtained,

$$\frac{\xi^2}{\xi^2 + \eta^2 + \zeta^2 + a^2 - \dfrac{P}{\mu}} + \frac{\eta^2}{\xi^2 + \eta^2 + \zeta^2 + b^2 - \dfrac{P}{\mu}} + \frac{\zeta^2}{\xi^2 + \eta^2 + \zeta^2 + c^2 - \dfrac{P}{\mu}} = 1 \quad (h),$$

determines three real values for P, which are the moments of inertia round principal axes through ($\xi\,\eta\,\zeta$).

Glasgow, Jan. 6, 1846.

POSTSCRIPT.

20. If in equation (h) of the last section, P have a given value, Π; and if x, y, z be any values of ξ, η, ζ which satisfy the equation in this case, the locus of $x\,y\,z$ will be a surface

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* In the first part of this paper (see p. 130, § 6 and Note) two ellipsoids were mentioned, either of which, different for different points of the body, may be described round any point as centre, and affords a geometrical construction for determining the principal axes of the body through this point. The *central ellipsoid* is unique in a solid body; its principal axes coincide in direction with the principal axes of the body through the centre of gravity; and the semiaxes are equal to $\sqrt{\frac{A}{\mu}}$, $\sqrt{\frac{B}{\mu}}$, $\sqrt{\frac{C}{\mu}}$ (radii of gyration), and thus its form, position, and magnitude, are entirely fixed in the body. The axes of either of the other ellipsoids for any point of the body coincide with the principal axes of the body through the point, and, arbitrary in absolute magnitude, are proportional to $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}, \frac{1}{\sqrt{C}}$, or to $\frac{1}{\sqrt{F}}, \frac{1}{\sqrt{G}}, \frac{1}{\sqrt{H}}$, the first being Poinsot's *momental ellipsoid*, and the second the ordinary *ellipsoid of construction*.

† This theorem has also been demonstrated by Mr. Townsend (Fellow of Trinity College, Dublin). His investigation, which is connected with the geometrical properties of confocal surfaces, and of enveloping cones, is entirely different from that given above, and will be published in an early Number of the *Journal*. I am informed by Mr. Townsend that another demonstration of the same theorem, which will also be contained in his paper, has been given (but not, so far as I am aware, published), several years ago, by Professor Maccullagh.

possessing the property that at each point one of the principal moments is equal to Π. Hence the equation of what may be called an *equimomental surface* is

$$\frac{x^2}{r^2 + a^2 - \frac{\Pi}{\mu}} + \frac{y^2}{r^2 + b^2 - \frac{\Pi}{\mu}} + \frac{z^2}{r^2 + c^2 - \frac{\Pi}{\mu}} = 1 \ldots\ldots (a).$$

By giving Π different values, we obtain different *conjugate* equimomental surfaces. The moment of inertia round any line not passing through the centre of gravity is greater than the moment round a line parallel to it through this point, and hence the smallest possible moment of inertia is that round the principal axis through the centre of gravity, of least moment. Hence by giving Π all values from μc^2 to ∞ (a, b, c are supposed to be in order of descending magnitude), we obtain all the possible equimomental surfaces for the body. These surfaces possess many remarkable properties, which derive additional interest from the circumstance that similar surfaces present themselves, with a very important signification, in the undulatory theory of light, Fresnel's *wave surface* in a biaxal crystal being the same as an *equimomental surface* in a solid body. The form of the equimomental surface is the same as that of the wave surface, when $\frac{P}{\mu}$ is greater than a^2, and is therefore well known in this case; but when $\frac{P}{\mu}$ is between a^2 and b^2, or b^2 and c^2, it will present many remarkable peculiarities, and may be an interesting subject for investigation.

21. The following theorem, suggested to me by a corresponding one in the undulatory theory, gives a construction for finding that principal axis through any point of the equimomental surface (a) round which the moment is Π.

If P be any point of an equimomental surface, and OQ a perpendicular from the centre to the tangent plane, then PQ is the principal axis round which the moment of inertia is Π.

To prove this, let l, m, n be the direction-cosines of the tangent plane at P, and let OQ, or $lx + my + nz, = v$. Then, denoting $\frac{\Pi}{\mu} - a^2$, $\frac{\Pi}{\mu} - b^2$, $\frac{\Pi}{\mu} - c^2$ by α, β, γ, and $\frac{x^2}{(r^2-\alpha)^2} + \frac{y^2}{(r^2-\beta)^2} + \frac{z^2}{(r^2-\gamma)^2}$ by S, we have

$$\frac{l}{\frac{x}{r^2-\alpha} - Sx} = \frac{m}{\frac{y}{r^2-\beta} - Sy} = \frac{n}{\frac{z}{r^2-\gamma} - Sz} \ldots . (b),$$

from which we deduce, by ordinary processes,

$$\frac{lx}{r^2-\alpha}+\frac{my}{r^2-\beta}+\frac{nz}{r^2-\gamma}=0\ldots\ldots\ldots(c),$$

and each member of equations (*b*) is

$$=\frac{v}{1-Sr^2},\quad \text{or}=\frac{1}{-Sv}.$$

Hence $$S(r^2-v^2)=1\ldots\ldots\ldots\ldots\ldots(d),$$

and we have

$$\left.\begin{aligned} lv\,(r^2-\alpha)&=x\,(v^2-\alpha)\\ mv\,(r^2-\beta)&=y\,(v^2-\beta)\\ nv\,(r^2-\gamma)&=z\,(v^2-\gamma)\end{aligned}\right\}\ldots\ldots\ldots(e).^*$$

Now lv, mv, nv are the coordinates of Q, and hence if λ, μ, ν be the direction-cosines of PQ, we have

$$\left.\begin{aligned} &\frac{\lambda}{r^2-\alpha}=\frac{\mu}{r^2-\beta}=\frac{\nu}{r^2-\gamma}\\ \text{or}\quad &\frac{\lambda}{r^2+a^2-\frac{\Pi}{\mu}}=\frac{\mu}{r^2+b^2-\frac{\Pi}{\mu}}=\frac{\nu}{r^2+c^2-\frac{\Pi}{\mu}}\end{aligned}\right\}\ldots\ldots(f),$$

which prove the theorem enunciated, since, by (*b*), § 18, these are the equations for determining the principal axis, corresponding to a root Π of the cubic equation which determines the three principal moments at any point $(x\,y\,z)$.

The locus of points on any surface

$$\frac{x^2}{a^2+\Theta}+\frac{y^2}{b^2+\Theta}+\frac{z^2}{c^2+\Theta}=1\ldots\ldots\ldots(g),$$

(confocal with the central ellipsoid) for which one of the principal moments is equal to Π, is determined by the equations (*a*) and (*g*) considered as simultaneous. By subtracting the former from the latter, we get

$$\left(r^2-\frac{\Pi}{\mu}-\Theta\right)\left\{\frac{x^2}{(a^2+\Theta)\left(r^2+a^2-\frac{\Pi}{\mu}\right)}+\frac{y^2}{(b^2+\Theta)\left(r^2+b^2-\frac{\Pi}{\mu}\right)}+\frac{z^2}{(c^2+\Theta)\left(r^2+c^2-\frac{\Pi}{\mu}\right)}\right\}=0.$$

* See *Math. Journal*, vol. I. (First Series) p. 8, or Gregory's *Examples*, p. 232, where the same formulæ occur in the investigation of the wave surface.

Hence the required locus on the surface (g) consists of two portions, given by the equations

$$r^2 = \frac{\Pi}{\mu} + \Theta \;\ldots\ldots\ldots\ldots\ldots (h),$$

$$\frac{x^2}{(a^2+\Theta)\left(r^2+a^2-\dfrac{\Pi}{\mu}\right)} + \frac{y^2}{(b^2+\Theta)\left(r^2+b^2-\dfrac{\Pi}{\mu}\right)} + \frac{z^2}{(c^2+\Theta)\left(r^2+c^2-\dfrac{\Pi}{\mu}\right)} = 1 \ldots (k),$$

taken separately. Now, by comparing (h) with equations (g) and (c') of § 19, we see that the moment of inertia round a normal at $x\,y\,z$ to the surface (g) is Π. Hence the first portion of the curve is the locus of points for which the moment of inertia round a normal to the surface (g) is constant. Also unless (h) be satisfied at any point, the moment of inertia round a normal is not Π, and hence (k) must represent the locus of points for which one of the principal moments round axes in the tangent plane is constant. Now these axes are, by what has been proved above, normals to the other two confocal surfaces through the same point, and are therefore tangents to the principal sections of the surface (g). Again, the principal axis round which the moment is Π, lies in the tangent plane to the equimomental surface through the point, corresponding to the moment Π; hence it must be the intersection of the tangent plane to this surface and the surface (g), and consequently this line of intersection must touch a principal section of (g). Hence the second portion of the curve on (g) is a line of curvature.* Thus we conclude that the equimomental surface (a) cuts the surface of the second order (g) in a "spherical conic" and a line of curvature. The moment of inertia round a tangent at any point of the latter curve and round a normal to the surface (g) at any point of the former is equal to Π. Also, since the principal axis round which the moment is Π, lies in the equimomental surface, it follows that, along the spherical conic, the equimomental surface cuts the surface of the second order at right angles, a result which may be easily verified.

* This theorem was communicated to me by Mr. Cayley before I had convinced myself of its truth by the proof given in the text. His demonstration is given below.

22 As an illustration of what has been proved in the last section, let us suppose $\Theta = 0$, so that (g) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

the equation of the central ellipsoid. It may be readily shewn that the moments of inertia round normals to this surface lie between the limits c^2 and a^2 (the mass being taken as unity), and those round lines touching it, between $b^2 + c^2$ and $a^2 + b^2$. Hence if Π have all values from c^2 to a^2 given it in succession, all the equimomental surfaces which intersect the central ellipsoid in a real spherical conic will be obtained; and if all values from $b^2 + c^2$ to $a^2 + b^2$ be given, all the equimomental surfaces which intersect the central ellipsoid in real lines of curvature will be obtained. The values of Π from $b^2 + c^2$ to $c^2 + a^2$ will give equimomental surfaces which cut the ellipsoid in the lines of curvature corresponding to the confocal hyperboloids of one sheet; and the other set of lines of curvature are therefore given by values of Π between $c^2 + a^2$ and $a^2 + b^2$.

Since $b^2 + c^2 > a^2$, except in the limiting case of the body being reduced to a portion of the plane of yz, when these quantities are equal, the values of Π which make one curve of intersection real must make the other imaginary. Hence the same equimomental surface can only cut the central ellipsoid in a "spherical conic" or a line of curvature, but not in both. The same may be shewn to be true for any confocal ellipsoid (but not for any of the hyperboloids). Hence any of the equimomental surfaces which cuts one of the ellipsoids must do so along either a spherical conic or a line of curvature, but not along both.

There are many interesting subjects of investigation relative to these equimomental surfaces which present themselves; such as the properties of the three equimomental surfaces which intersect in a given point and of the curves along which they cut one another, the forms of the different classes of equimomental surfaces, and the remarkable properties of principal axes at different points (corresponding to the *conical refraction* of light), which are due to the singular points of these surfaces. These may be discussed in a future communication; but the length to which this paper has already been extended, prevents me from going farther in the subject at present.

St. Peter's College, May 8, 1846.

NOTE ON A GEOMETRICAL THEOREM CONTAINED IN THE PRECEDING PAPER.

By Arthur Cayley.

It is easily shown that if three confocal surfaces of the second order pass through a point P, then the square of the distance of this point from the origin is equal to the sum of the squares of three of the axes, no two of which are parallel or belong to the same surface (the squares of one or two of the axes of the hyperboloids being considered negative); *i.e.* if

$$\frac{x^2}{a^2+h}+\frac{y^2}{b^2+h}+\frac{z^2}{c^2+h}=1,$$

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1,$$

$$\frac{x^2}{a^2+l}+\frac{y^2}{b^2+l}+\frac{z^2}{c^2+l}=1;$$

then $$x^2+y^2+z^2=a^2+b^2+c^2+h+k+l.$$

In fact these equations give

$$x^2=\frac{(a^2+h)(a^2+k)(a^2+l)}{(a^2-b^2)(a^2-c^2)},$$

$$y^2=\frac{(b^2+h)(b^2+k)(b^2+l)}{(b^2-a^2)(b^2-c^2)},$$

$$z^2=\frac{(c^2+h)(c^2+k)(c^2+l)}{(c^2-a^2)(c^2-b^2)}.$$

And adding these and reducing, we have the relation in question; which is also immediately obtained by forming the cubic whose roots are h, k, l.

From this property may be deduced the theorem given by Mr. Thomson in the preceding memoir. In fact, writing

$$r^2=x^2+y^2+z^2,$$

and $$k=-a^2-b^2-c^2+h,$$

we see that in consequence of these relations the equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

$$\frac{x^2}{r^2+a^2-h}+\frac{y^2}{r^2+b^2-h}+\frac{z^2}{r^2+c^2-h}=1,$$

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1,$$

are equivalent to two independent equations, *i.e.* the third can be deduced from the two first. Now the first equation is that of an ellipsoid (or generally a surface of the second order, since a, b, c are not necessarily real). The second is that of what may be called a conjugate equimomental surface, defining the term as follows: "The conjugate equimomental surfaces of an ellipsoid (or surface of the second order) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, are the equimomental surfaces derived in the usual manner from any surface of the second order $\frac{x^2}{h - a^2} + \frac{y^2}{h - b^2} + \frac{z^2}{h - c^2} = 1$, which is confocal with the *conjugate* surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$ of the given ellipsoid," viz. by measuring along any line through the centre distances equal to the axes of the section by a plane through the centre perpendicular to this line, and taking the locus of the points so determined for the equimomental surface. The third equation is that of a surface confocal with the given ellipsoid; hence the theorem, "The curves of curvature of a given ellipsoid lie upon a system of conjugate equimomental surfaces."

But since the first and second equations are evidently satisfied by the combination of the first equation with the relation $r^2 = h$, which is that of a sphere, we have also, "The curve of intersection of the ellipsoid with any one of the conjugate equimomental surfaces, is composed of the line of curvature, and of a spherical conic." And these two curves being each of them of the fourth order make up the complete curve of intersection, which should obviously be of the eighth order.

It would be an interesting question to determine the relations existing between the curve of curvature and the spherical conic, which have been thus brought into connection by means of the conjugate equimomental surfaces; *i.e.* between the two curves obtained by combining the equation of the ellipsoid with

$$\frac{x^2}{a^2 + k} + \frac{y^2}{b^2 + k} + \frac{z^2}{c^2 + k} = 1,$$

$$r^2 = a^2 + b^2 + c^2 + k,$$

respectively: but it will be sufficient at present to have suggested the problem.

ON PRINCIPAL AXES OF A BODY, THEIR MOMENTS OF INERTIA, AND DISTRIBUTION IN SPACE.

By RICHARD TOWNSEND, Fellow of Trinity College, Dublin.

1. As the centre of gravity of a body or system of bodies may be defined either by its analytical or by its fundamental physical property, and as it may be readily shewn that either follows from the other, so may we define a principal axis at any point of a solid body either by its analytical or by its dynamical property with respect to that point, and these properties may be easily proved to be each a necessary consequence of the other. This is usually done as follows.

Let O be any point, and OZ any axis drawn therethrough, round which suppose the body to revolve with an angular velocity ω; then will every element dm have a centrifugal force impelling it perpendicularly out from the axis, and equal to $\omega^2 rdm$, r being the distance of dm from OZ.

Transferring all these forces to O, each by a parallel movement, we shall have a number of forces passing through O and lying all in a plane perpendicular to OZ, and a number of moments in planes passing all through OZ; the forces will compound a single force (which may $= 0$) at right angles to the axis, and the pairs a single pair in a plane passing therethrough.

To find the values of the resultant force and of the resultant moment, let each component force $\omega^2 rdm$ be resolved into two $\omega^2 xdm$, $\omega^2 ydm$ along any two fixed axes OX, OY at right angles to OZ, and therefore in the plane which contains the whole system of forces; and let each component moment $\omega^2 rzdm$ be resolved into two $\omega^2 xzdm$, $\omega^2 yzdm$ in the planes ZOX, ZOY: the sums of the resolved parts of all these forces along each axis, and of all the pairs in each plane, will be then equal to the resolved parts of their respective resultants along the same axes and in the same planes.

The quantities $\omega^2.\Sigma xdm$ and $\omega^2.\Sigma ydm$ are, therefore, the components of the resultant force along OX and OY, and the quantities $\omega^2.\Sigma zxdm$ and $\omega^2.\Sigma zydm$ are those of the resultant moment in the planes ZX and ZY.

Now, if any number of forces be in equilibrium, their resultant moment with respect to any point whatsoever must vanish, and if they compound a single resultant force, their resultant moment with respect to any point through which that force passes must vanish; and conversely, if at any point the resultant moment, or, which is the same thing, the re-

solved parts of that resultant in any directions be equal to nothing; then will the forces either equilibrate or compound a single resultant passing through that point.

If, therefore, OZ be a *permanent* axis of rotation (the point O of the body alone being fixed), then must the quantities $\Sigma xzdm$ and $\Sigma yzdm$ be $= 0$ for *every* pair of coordinate axes through O at right angles to OZ, and conversely. If these analytical quantities be both $= 0$, then for the point O will the resultant moment of the centrifugal forces vanish, or the axis be permanent.

2. The forces will not equilibrate except in the particular case when the point is the centre of gravity; for, the components of their resultant along OX and OY are respectively $\omega^2\Sigma xdm$ and $\omega^2\Sigma ydm = \omega^2\bar{x}M$ and $\omega^2\bar{y}M$.

Hence, an axis which is principal for one point of a body is principal for no other point on itself, and therefore for no other point whatever, except in the particular case of the centre of gravity whose principal axes are principal for all points along them. This is evident, for when any number of forces compound a single resultant, their resultant moment with respect to an assumed point will vanish only when that point is on the resultant, except in the particular case when that resultant is nothing, that is, when the forces are in equilibrium.

Again, since in general systems of forces neither equilibrate nor compound a single resultant, it follows that an axis taken at random in a body may not be principal for any point at all; and that such is often actually the case, appears at once from the values for the components of the resultant force and moment, which for *every* point on the axis must be such that $\Sigma xzdm : \Sigma yzdm :: \Sigma xdm : \Sigma ydm$, in order that the resultants of the two systems of parallel forces, $\omega^2 xdm$ in the plane of xz and $\omega^2 ydm$ in the plane of yz, which both pass through the axis OZ, should meet it at the same point, and therefore compound a single resultant: but this proportion does not in general hold.

Hence, though the number of principal axes in a body may perhaps be infinite, it may still be small in comparison with the number of axes which are not principal.

3. If on every line diverging from any point O of a solid body we measure off a portion r, such that the square of its reciprocal multiplied by the mass of the body shall equal the moment of inertia round it; then, as is well known, will the locus of the extremity of r be an ellipsoid (the "Momental" Ellipsoid of Poinsot). This is usually shewn as follows.

Assuming arbitrarily any three rectangular axes through O, let $A'\,B'\,C'$ be equal to the moments of inertia round them, and let $A''\,B''\,C''$ be equal to the quantities $\Sigma yzdm$, $\Sigma zxdm$, and $\Sigma xydm$ with respect to the same; then will the moment of inertia round the line whose direction-angles are $\alpha\,\beta\,\gamma$ be equal to

$$A'\cos^2\alpha + B'\cos^2\beta + C'\cos^2\gamma - 2A''\cos\beta\cos\gamma - 2B''\cos\gamma\cos\alpha - 2C''\cos\alpha\cos\beta = \frac{M}{r^2}.$$

Multiplying this by r^2, we get the equation of the surface in question, viz.

$$A'x^2 + B'y^2 + C'z^2 - 2A''yz - 2B''zx - 2C''xy = M \; . \; . \; \text{(I)},$$

a central surface of the second order, which, as moments of inertia are essentially positive, is therefore an ellipsoid referred to its centre.*

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* M. Poinsot, in his elegant tract on Rotation, has denominated the above ellipsoid at any point of a body, the "central" ellipsoid of that point, and that name had been at first used in this paper, but having been found very inconvenient, in consequence of the *centre* of gravity appearing in almost every enquiry, and, as such, having no connexion with the ellipsoid at any other point, the term "momental" (for which there is high authority) has been since adopted in preference.

If round O as centre we conceive a sphere to be described with radius $=1$, and that the polar plane of every point of the momental ellipsoid be taken with respect to that sphere, then will the envelope of all these planes be another ellipsoid, which, as is evident, will be concentric and coaxal with the former, and which, as obviously possessing the property that the squares of the perpendiculars from O upon all its tangent planes multiplied each by the mass of the body are equal to their moments of inertia, may be called the *ellipsoid of inertia* of that point. These two concentric ellipsoids, *spheropolars reciprocal* to each other, and thus intimately connected at each individual point of a body, are of constant occurrence in all enquiries respecting the equilibrium and motion of solid bodies, and should be familiarly known and always considered together, both being important and possessing their advantages each over the other. If we know either for any point of a body we of course have the other for the same point, but they both obviously vary, in magnitude, position, and figure, from one point of a body to another.

The *particular* ellipsoid of inertia round the centre of gravity (which is a very important surface in all that relates to the present subject) has been called (for an obvious reason) the *ellipsoid of gyration*, and is the ellipsoid which Mr. Thomson, Fellow of St. Peter's College, has distinguished by the shorter and more expressive name of *the central ellipsoid*.

The surface locus of feet of perpendiculars from any point O, upon all the tangent planes to its ellipsoid of inertia, that is, the surface round any point of a body the squares of whose radii multiplied each by the mass of the body are equal to their moments of inertia, may be called the *surface of inertia* of that point. We shall have occasion as we proceed to notice it also; it varies like the others from point to point of every body, is always of the fourth order, has two circular sections passing both through its mean axis, and coincident with those of the momental ellipsoid, and is the same in shape as the well-known "surface of elasticity" in the Wave Theory of Light.

For each point O, this ellipsoid is, from its very nature, fixed in magnitude and position with respect to the body, and is of course independent of the arbitrarily assumed directions of the coordinate axes; though if the latter be changed, the coefficients $A'\, B'\, C'\, A''\, B''\, C''$ in the new equation will no longer represent the *same* sums as before, but will be equal to the similar sums with respect to the new axes. Conversely, therefore, this ellipsoid once determined gives us the values of these six integrals for every rectangular system of coordinate axes assumed at pleasure through O.

Now, in *every* ellipsoid (and therefore in the above) there exist *three* lines, passing through its centre and at right angles to each other (and generally but three), to which as axes of coordinates if the surface be referred, the coefficients $A''\, B''\, C''$ will all disappear from the equation, and for each of which, whatever be the directions of the other two coordinate axes, provided they be both at right angles to the first, two of the same quantities will vanish. This, therefore, being the analytical property of a principal axis, proves that for every point of a solid body there are, at least, three principal axes at right angles to each other, the axes, viz., of its momental ellipsoid.*

4. By means of the momental ellipsoid we may also arrive at the same result from the dynamical property of a principal axis.

For, suppose the body to revolve round any diameter $2r$ of that ellipsoid, with which, as the coordinate rectangular axes are quite arbitrary, let one of them, that of z, coincide; then, in the equation of the surface (3),

$$A'x^2 + B'y^2 + C'z^2 - 2A''yz - 2B''zx - 2C''xy = M,$$

will the quantities A'' and B'', multiplied each by the square of the angular velocity, be equal to the components in the planes of zy and of zx respectively of the resultant moment of all the centrifugal forces.

A'' and B'' are, therefore, proportional to the cosines of the angles which the plane of that resultant passing through the axis of z makes with the planes of zy and of zx respectively, and their *signs* are the same as those of the *directions* in which the components tend to draw the axis of rotation.

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* This proof is due to Professor MacCullagh, and was given by him at his mathematical lectures in Trinity College, Dublin.

At the point where that axis meets the surface, let now a tangent plane be drawn, then (the coordinates of that point being $x = 0$, $y = 0$, $z = r = \frac{\sqrt{M}}{\sqrt{C'}}$) will its equation be

$$-B''r.x - A''r.y + C'r.z = M \ldots\ldots\ldots (\text{II}),$$

from which we see that the perpendicular p, let fall from the centre on this tangent plane, makes with the axes of x and of y, respectively, angles whose cosines are proportional to the quantities $-B''$ and $-A''$, and therefore proportional to the components of the resultant moment with their signs changed, that is, with their directions changed into the opposite.

If, therefore, a body revolve round any radius of its momental ellipsoid at any point, the plane of the resultant centrifugal moment will be that of the radius and perpendicular, and its tendency will be to draw the axis of rotation *from* the perpendicular; theorems which are due to Poinsot and to Professor Maccullagh.

Again, from (II) we have

$$p^2.(B''^2 + A''^2 + C''^2).r^2 = M^2;$$

from which, since $C''^2 = \frac{M^2}{r^4}$, we get

$$B''^2 + A''^2 = \frac{M^2}{p^2r^2} - \frac{M^2}{r^4} = \frac{M^2}{r^2}\left(\frac{1}{p^2} - \frac{1}{r^2}\right) = M^2p'^2(r'^2 - p'^2) \ldots (\text{III}),$$

by changing at every point of the ellipsoid r and p into their reciprocals p' and r', p' being supposed $= \frac{1}{r}$ and $r' = \frac{1}{p}$.

If, therefore, a body revolve with the same angular velocity round different radii of its momental ellipsoid at any point, then will the magnitude of the resultant centrifugal moment be proportional to the area of the right-angled triangle $p'r'$.*

Now, in every ellipsoid there exist *three* diameters at right angles to each other (the axes of the surface), for which p will coincide with r, and therefore p' with r', and for which,

* Translating the above properties to the *ellipsoid of inertia*, we, hence, know that—If a body revolve round a perpendicular on a tangent plane to its ellipsoid of inertia at any point, then will the plane of the resultant centrifugal moment be that of the perpendicular p' and of the corresponding radius r', its direction will be *from* r′ *towards* p′, and its magnitude (for an invariable angular velocity) will be proportional to the area of the right-angled triangle $p'r'$, being in all cases equal to the quantity $M.\omega^2$. (double that area).

consequently, the area of the triangle $p'r'$ will vanish. If, therefore, a body revolve at any point round either of these diameters of its momental ellipsoid, the resultant centrifugal moment with respect to that point will vanish; which proves that at every point of a body the axes of its momental ellipsoid are principal axes.

5. At any point O let $A\ B\ C$ be the three principal moments of inertia; then, referring the momental ellipsoid to the principal axes, its equation will be

$$Ax^2 + By^2 + Cz^2 = M \ldots\ldots\ldots\ldots \text{(IV)}:$$

from which (since all lines which are axes of that surface, and none others, are principal axes at O) it appears that, to know the number of principal axes which a body admits of at a given point, we need only know the relative magnitudes of $A\ B$ and C for that point.

There are, therefore, three varieties of points in a body with respect to the number of principal axes which they admit of.

The most frequent case is, when A, B, and C are all unequal; such points admit of but three principal axes, viz. those of the ellipsoid. Secondly, two of them may be equal, in which case the ellipsoid will be of revolution and, besides the one perpendicular thereto, will have an infinite number of axes in the plane of its equator. Such points admit, therefore, of an infinite number of principal axes in the plane which contains the two of equal moment: these axes will moreover be all *equimomental*, for, their plane intersecting the ellipsoid in a circle, the radii of that surface with which they coincide, and therefore their moments of inertia will be all equal, (3). Thirdly, A, B, and C may be all equal; the ellipsoid will then be a sphere, and *all* axes for such points will be principal, and also equimomental.

(In *every* body there exists, as we shall see, a curve locus of points of the second species; but if a point of the third species exist, either it must be the centre of gravity, or else that centre must be a point of the second species where the two *equal* principal moments are each *less than* the third; in which latter case there will be two such points, both on the third central principal axis, at opposite sides of the centre of gravity, and equidistant therefrom by the interval $\pm\sqrt{\left(\frac{A-C}{M}\right)}$, A and C being the unequal principal moments, and M the mass of the body).

6. If, in the general case, the momental ellipsoid at any point O be intersected by any concentric sphere, then will all axes through O which pass through the curve of intersection be equimomental.

This is obvious, for their intercepts between the centre and the curve, being radii of a sphere, are all equal to each other; and being radii of the momental ellipsoid, are equal to the inverse square roots of their moments of inertia.

Let I be the moment round a system of such axes, then will $I.(x^2+y^2+z^2)=M$ be the equation of the sphere; subtracting this from that of the ellipsoid $Ax^2+By^2+Cz^2=M$, we get that of the cone which the axes generate, viz.

$$(A-I)\,x^2+(B-I)\,y^2+(C-I)\,z^2=0\,.\,.\,.\,.\,.\,(\mathrm{V}).$$

At every point, therefore, of a body, every system of equimomental axes generates a cone of the second order.

Each point of the body has, of course, an infinite number of such cones; for I may have all values between the limits A and C: and from the above equation it appears that the cones of each system have all the same principal axes, viz. those of the body at their common vertex.

(From this it appears, that if at any point of a body we could find one of its cones of equimomental axes, we should have an obvious geometric construction by which to determine the principal axes for that point).

The cones of each system have also all the same cyclic planes, viz. those of the momental ellipsoid.

For, the quantity $\pm\sqrt{\left\{\dfrac{(A-I)-(B-I)}{(B-I)-(C-I)}\right\}}$, which expresses the tangent of the angle that the cyclic planes (which here pass all through the axis of y) make with the plane of xy, is independent of I, and $=\pm\sqrt{\left(\dfrac{A-B}{B-C}\right)}$, which last expresses the tangent of the similar angle in the momental ellipsoid.

When $I=B$, the cone (V) degenerates into two planes passing through the axis of y: all axes, therefore, at any point of a body for which the moment of inertia equals the mean principal moment, lie in two planes, equally inclined to the axis of least, and therefore of greatest, moment; viz. the cyclic planes of the momental ellipsoid.

If the ellipsoid be of revolution, so will also the whole system of equimomental cones; the cyclic planes will also coincide in one which will be the limit to the system of cones, and we get the property already noticed, that when two of

the principal moments are equal, all axes in their plane will be equimomental.

Hence, conversely, if at any point of a body one of the cones of equimomental axes be a cone of revolution, such point will admit of an infinite number of principal axes in the principal plane perpendicular to the internal axis of the cone.

If the ellipsoid be a sphere, the cones will be all indeterminate, as they ought to be.

7. In the particular case when the point O is taken off at infinity, the reasonings by which the conclusions of the preceding section were established, of course fail, the momental ellipsoid being there infinitely slender; but (as is usual in all such cases) the results are then simpler. It should be mentioned, however, lest an erroneous conclusion might be hastily drawn, that some of the latter *appear* to be at variance with others which have been obtained in the general case.

For instance, from the well-known and very important property of the centre of gravity with respect to the moments of inertia round parallel axes, it appears that every system of equimomental axes which are all parallel to each other in any direction whatever, will generate a cylinder of revolution round the parallel axis through the centre of gravity; and, by giving different values to the moment of inertia, that the whole system of equimomental cylinders will be therefore all co-axal. But these cylinders (since their sides, like every other system of parallel lines, pass all through an infinitely distant point) are cones of the second order having a common vertex at infinity, and from this we might hastily conclude that *every* point at infinity admits of an infinite number of principal axes, since for every such point the system of equimomental cones is of revolution.

But that is not the case, for (as we shall presently see) the principal axes of a body are as accurately determinable for a point at infinity as for any other point; and we shall then also see from the equation of the equimomental system of cones, which we can find for any point, why it is that in the above particular case they are of revolution, while at the same time their three axes are all fixed. For the present we will only remark, that should an ellipse, variable, in magnitude, position, and figure, according to any law, and whose axis, determinable in each case from that law, enjoy some property peculiar to themselves, pass through the particular figure of a circle, we have obviously no right to extend to *any* pair of diameters of that circle the peculiar property of

the axes, but must take *the particular pair* determined by the law which all the other axes follow. Such is actually the case in the present instance, and instances of a similar nature are of frequent occurrence, as, for example, in the determinable points of intersection of two consecutive curves which ultimately coincide. The above, therefore, presents no real difficulty, and has only been noticed to prevent the chance of an erroneous conclusion.

8. At every point of a body the sum of the moments of inertia round any three rectanglar axes is equal to the sum of the three principal moments.

For, in every ellipsoid, and therefore in the momental ellipsoid at every point of a body, the sum of the squared reciprocals of any three rectangular semidiameters is equal to the sum of the squared reciprocals of the three semiaxes.

If, therefore, round the centre of gravity of a body as centre, a sphere be described, then for all points of that sphere will the sum of the three principal moments be constant (a property of the centre of gravity which may be considered as an extension of its ordinary property respecting equidistant axes).

For that sum at every point of the sphere is equal to the sum of the moments round three parallel axes through the centre + Mr^2, M being the mass of the body and r the radius of the sphere.

If at any point of a body a number of axes be drawn all in the same plane, then will the sum of the moments round every two at right angles to each other be constant.

For that sum, together with the moment round the axis perpendicular to the plane is constant, and the latter moment is the same for all.

If the plane be fixed, the locus of the points therein, for which this sum, constant for each, will have the same value, will be a circle of which the centre will be the foot of the perpendicular dropped on the plane from the centre of gravity.

For perpendiculars to the plane erected at every point of such a circle will be all equimomental axes (7), and also the sum of the moments round every three rectangular axes, constant for each of these points, will be the same for them all, since they lie on a sphere whose centre is the centre of gravity.

If a system of equimomental axes lie all in a plane passing through the centre of gravity, they will envelope an ellipse (or hyperbola) of which that point will be the centre.

To shew this, let I' I'' be the moments of inertia round the semiaxes r' r'' of the ellipse in which the plane intersects the momental ellipsoid at the centre of gravity, ρ' ρ'' the radii of the same parallel and perpendicular to one of the equimomental axes, d the distance between that axis and ρ', and α β the angles which ρ'' or d makes with r' and r''; then, denoting by I the constant moment common to all the axes, we shall have $\frac{M}{\rho^2} + M.d^2 = I$, from which, since $\frac{1}{\rho^2} = \frac{1}{r'^2}\cos^2\alpha + \frac{1}{r''^2}\sin^2\alpha$, $\frac{M}{\rho^2} = I'\cos^2\alpha + I''\sin^2\alpha$, and we get

$$M.d^2 = (I - I')\cos^2\alpha + (I - I'')\sin^2\alpha \; \ldots\ldots \; (a):$$

the equimomental axes therefore all envelope a central conic, the squares of whose semi-axes are $\frac{I - I'}{M}$ and $\frac{I - I''}{M}$, and which will be therefore an ellipse or hyperbola, as the case may be.

For different systems of equimomental axes in the same plane I will vary but I' and I'', and their axes will remain the same: hence we know that the system of conics enveloped by all the systems of axes will be coaxal and confocal, their common axes being those of the section of the central ellipsoid.

The ellipse a', b', or $\sqrt{\frac{M}{I'}}$, $\sqrt{\frac{M}{I''}}$, with which the conics are all confocal, is obviously the ellipse cyclo-polar reciprocal* to the ellipse section of the momental ellipsoid by the plane of the axes, the radius of the reciprocating circle being 1, and the pole being the centre of that ellipse, or the centre of gravity.

9. Now, if a system of equimomental axes, lying all in any plane whatever, be projected orthographically upon a parallel plane through the centre of gravity (and therefore upon *any* parallel plane), the projections will be also a system of equimomental axes.

For the moment round each axis will exceed that round its projection by $M\delta^2$, δ being the distance between the planes.

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* The terms *cyclo* and *sphero* polar reciprocal, were first introduced by Mr. Ingram, Fellow of Trinity College, Dublin, in order to distinguish from the general class of polars with respect to *any* curve or surface of the *second* order (which class alone possesses the important property of *reciprocality*) that particular class where the reciprocating curve or surface is a *circle* or *sphere*.

Hence, from the above, we know that every system of equimomental axes which lie all in a plane will envelope a conic, an ellipse, or hyperbola, as the case may be. And, by giving different values to the moment of inertia, the whole system of conics thus enveloped will be all concentric, coaxal, and confocal; the common centre, axes, and foci, being those of the ellipse contour of the orthographic projection on that plane of the ellipsoid of inertia round the centre of gravity, that ellipsoid being the sphero-polar reciprocal of the momental ellipsoid round the same point to the concentric sphere $x^2 + y^2 + z^2 = 1$.

In the particular case, when the plane of the equimomental axes is parallel to either cyclic plane of the central ellipsoid, the conics will be obviously all concentric circles. But of this, as well as of the general case, more hereafter.

Any two of these conics in any plane being confocal, the locus of the intersection of a tangent to either with a rectangular tangent to the other, will be a circle concentric with both: hence we arrive at the property, already noticed, that the locus of points in any plane for which rectangular axes in that plane have a constant sum of moments, is a circle of which the centre is the projection on the plane of the centre of gravity.

If we have one of these conics in any plane, we may at once find the axes of maximum and of minimum moment in that plane for every point thereon; for we have only to draw from the point two tangents to the conic, and then will the two rectangular lines which bisect the angles between them be the axes required: or, which comes to the same, we may describe through the point the ellipse and hyperbola confocal to the conic, and then will their normals at the point be the axes sought.

Since the tangents drawn from any point in the plane of a conic make equal angles with the tangents drawn from the same point to any confocal conic, a still easier construction than either of the above would be to connect the point with the foci of the given conic, and to bisect the angles between the connecting lines; the bisecting lines would then be the axes sought.

10. Now the axes so found at any point in a given plane coincide obviously with the geometric axes of the section in which the plane intersects the momental ellipsoid of that point; the above therefore affords an easy means of finding the axes of every central section of that ellipsoid at any point of a body.

It also enables us to find the locus of points for which these axes in a given plane enjoy a given property: for instance, let it be required to find the locus of points in the plane for which they remain parallel to each other.

Connecting all the points with the two foci of the system of conics in that plane, we shall have a series of triangles having all the same base, viz. the line joining the foci, and of which the bisectors of the internal and external vertical angles are all parallel to two given rectangular lines: the base, therefore, and the difference of the base angles being the same for all these triangles, the locus of their vertices (that is, of the points in question) will be an equilateral hyperbola, passing through the two foci, and having for centre the middle point of the line joining the same; that is, the projection of the centre of gravity on the given plane, and for asymptotes, lines parallel to the given bisectors.

In every plane, therefore, taken at random in a body, there exists a series of equilateral hyperbolas, such that for all the points of each the momental ellipsoid will intersect the plane in ellipses whose axes will be all parallel to each other, and these hyperbolas will have all the same centre, viz. the projection on the plane of the centre of gravity, they will all pass through two fixed points, viz. the foci of the system of confocal conics which envelope the systems of equimomental axes in that plane, and the locus of their vertices will be the lemniscata of the principal hyperbola, that, viz., whose axis is the line joining the two foci.

Since perpendiculars to the plane erected at the centre and foci of its system of conics pass through the centres and foci of the systems of conics in every parallel plane (9), it appears that if we move the above plane parallel to itself, each of the hyperbolas will describe a cylinder of the second order. Hence we know that the locus of those points in a body, for which two rectangular radii of their momental ellipsoids drawn in directions parallel to two given lines are the axes of the central sections whose planes they determine, will be a surface of the second order, a cylinder whose base will be an equilateral hyperbola in the plane parallel to the two given directions, whose axis perpendicular to that plane will pass through the centre of gravity, and whose asymptotic planes passing both through that axis will be parallel each to one of the given directions.

If in any plane we take the particular conic of its confocal system of equimomental envelopes which passes through the two foci, that is, the infinitely flat ellipse or hyperbola which

constitutes the transition state from one of these species to the other, then will all axes in the plane which pass through either of these foci be tangents to that particular conic, and be therefore equimomental. Hence we know that in every plane taken at random in a body there exist two points (the foci of a certain ellipse (9),) for which all axes in that plane will be equimomental, and for every system of parallel planes the loci of the points will be two parallel right lines perpendicular to the system of planes, and equidistant from, and in a plane passing through, the centre of gravity.

Now if at any point of a body a system of equimomental axes lie all in a plane, that plane must be a cyclic plane of the momental ellipsoid at that point (6). Hence, from the above, we know that in every plane assumed at pleasure in a body, there exist two points for which the momental ellipsoid will intersect it in a circle. These points are easily found, being the foci of an ellipse contour of the projection on the plane of the ellipsoid of inertia round the centre of gravity (9), from which we see that their loci for every system of parallel planes will be two parallel right lines, viz. the focal lines of the projecting cylinder.

In the particular case when the plane is parallel to a cyclic plane of the momental ellipsoid at the centre of gravity, there will exist but one such point, viz. the projection of that centre: this is evident, for the projecting cylinder will then be of revolution, and its axis passing through the centre of gravity will intersect the plane in a single point, and will be the locus of the similar points for the system of parallel planes.

The centre of gravity, in this case, is itself the point in the particular plane passing therethrough; that plane intersects the cylinder of revolution (which, as we shall see, is an important surface) in a circle, the squares of whose radii, multiplied each by the mass of the body, are equal to their moments of inertia; and if at all the other points we describe in their own planes, circles possessing the same property, the whole system of circles thus described will obviously generate a surface of revolution round the axis of the cylinder: this it is easy to see will be an hyperboloid of one sheet touching the cylinder along its principal section through the centre of gravity.

For, let r be the radius of the cylinder, x the distance of one of the planes from the centre of gravity, and y the radius of the circle in that plane, then have we $My^2 = Mr^2 + Mx^2$, and therefore $y^2 - x^2 = r^2$, an equilateral hyperbola.

More generally, if at every point of any axis through the centre of gravity perpendiculars be erected all round the axis, and that portions be taken on each whose squares, multiplied each by the mass of the body, shall equal their moments of inertia; then, for the same reason as above, will the locus of the extremities of these portions be a surface all whose sections by planes through the axis will be equilateral hyperbolas.

(It is, perhaps, needless to say that this surface will not be an hyperboloid, for, except in the particular case just noticed, its sections perpendicular to the axis will not be curves of the second order. See Note Art. 3.)

If at all points of any plane drawn at will through the centre of gravity of a body, we erect perpendiculars whose squares multiplied each by the mass shall equal their moments of inertia, the locus of their extremities will be an hyperboloid of two sheets of revolution round the perpendicular through the centre of gravity.

For, drawing through that perpendicular any two rectangular planes, let the coordinates referred thereto of the extremity of one of the variable perpendiculars z be xy, then have we $Mz^2 = Mz_1^2 + M(x^2+y^2)$, and therefore $x^2+y^2-z^2=z_1^2$; the meridians therefore are equilateral hyperbolas, and the surface is of revolution round their transverse axis.

(The plane obviously need not pass through the centre of gravity, and the only difference will be that the centre of the hyperboloid will not be that centre, but its projection on the plane.)

For every plane through the centre of gravity we have a different hyperboloid; these are all connected by the property that the squares of their axes represent their moments of inertia. If, therefore, at the vertices of each we draw tangent planes, the whole system of planes thus produced will envelope an ellipsoid (of which we have spoken and shall frequently use again), the ellipsoid of gyration, spheropolar reciprocal with respect to the sphere $x^2+y^2+z^2=1$ of the momental ellipsoid at the centre of gravity. This is evident, for the coincident radii of the latter surface are inversely as the axes of the hyperboloids, shewing that the extremities of these radii are the poles with respect to that sphere of the tangent planes in question (Note Art. 3), and that, conversely, the locus of the vertices of the polar reciprocal system of hyperboloids is the momental ellipsoid at the centre of gravity.

Since every plane parallel to the axis of any one of these hyperboloids intersects it in an equilateral hyperbola, of which the centre is the projection on the plane of the centre of gravity, it appears that, if in any plane taken at will we draw any line whatever, and at all its points erect perpendiculars in the plane whose squares multiplied by M shall equal their moments of inertia, the locus of the extremities of those ordinates will be an equilateral hyperbola of which the line will be the imaginary axis, and whose centre will be the projection on that line of the centre of gravity.

If in place of drawing the ordinates perpendicularly to the assumed line, we draw them all obliquely to the same, the locus will be still an hyperbola, but its centre will no longer be the same as before, being in all cases the foot of the particular ordinate drawn through the projection on the plane of the centre of gravity, and that ordinate with the assumed line will be always conjugate diameters of the hyperbola.

11. Throughout the preceding section we had frequent opportunities of observing the symmetrical arrangement on every arbitrary plane round the projection thereon of the centre of gravity, of every system of axes whose moments of inertia follow all the same law, that point not only being a centre round which every thing is the same in opposite directions, but also the projected axes of the parallel central section of the momental ellipsoid at the centre of gravity dividing the plane into four regions, in which the axes of equal moment are arranged symmetrically and similarly round that centre, and the same being, of course, true for the centre of gravity itself with respect to every plane passing therethrough. This, however, we might have easily anticipated; for, from its property respecting the moments of inertia of parallel axes, we readily see that, more generally, the centre of gravity of every body possesses the same properties in *space*, everything being the same with respect to the moments of inertia of all axes similarly situated in opposite directions from that point, and from the additional circumstance that the surface round the same centre whose radii represent their moments of inertia is symmetrical and equal in the eight regions of space determined by the three central principal planes: we see also that everything is the same respecting the moments of inertia of axes and of systems of axes which are symmetrically situated in these eight different regions with respect to the three principal axes through the centre of gravity.

With respect to *principal* axes the same important properties tell us—

(1) That at every two points of a body equidistant, and in opposite directions from the centre of gravity, the principal axes are all mutually parallel to each other and respectively equimomental.

For, the moments of inertia being equal round every pair of parallel lines through any two such points, their momental ellipsoids will be equal, similar, and similarly placed; the axes, therefore, of these ellipsoids (with which the principal axes of the body at the same points coincide) must be parallel, and their moments of inertia respectively equal.

(2) That at every eight points of a body equidistant from, and similarly and symmetrically situated with respect to the three principal axes at the centre of gravity, the principal axes will be also all similarly and symmetrically situated with respect to the same three axes; which is evident from the symmetry of the momental ellipsoid round that centre.

Whatever, therefore, be the geometric distribution in space of the principal axes of a body fixed in position, we see that the centre of gravity is a point round which they are similarly and symmetrically situated in the eight regions of space determined by the three principal planes of that centre; a circumstance which will, of course, be verified when we ascertain the laws (whatever they be) which determine their directions at different points, but which, when we consider the dynamical property of these axes and the utter irregularity of almost all bodies with respect to figure, density, and structure, is by no means evident *à priori*.*

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* The symmetry here noticed holds also in many other cases in which it would be still more difficult, or rather impossible, to perceive, *à priori*, its necessary existence. For instance, in the highly elastic and absolutely incompressible ether from the vibrations of which arise the phenomena of light, the discoveries of Professor MacCullagh have shewn that at every point there exist three axes at right angles to each other with respect to which everything is the same in every eight directions of symmetry. Also, in the altogether different case of solid elastic bodies, the recent researches of Mr. Haughton, Fellow of Trinity College, Dublin, respecting their equilibrium and motion, have indicated that, whatever be their internal structure, there exist also at every point three rectangular axes round which a symmetry of a remarkable and unexpected nature is always found. It is a circumstance not a little surprising, that strict mathematical reasoning, setting out from the simple elementary principles which characterise each case, should be able to prove, in any case, the existence of this symmetry, which is a physical fact not apparently at all connected with the principles from which we set out. It was to call attention to this circumstance that we have made the above remark, lest in more difficult cases it might be conceived to be impossible to prove such existences.

12. Equimomental axes, in a body considered merely as such with no other restriction, are, of course, infinite in number, being so for each individual point: they are all, however, confined within very narrow limits round the centre of gravity, passing all through the space between two spheres which may be readily found.

For, let I be their common moment, and A and C the greatest and least central moments; then describing round the centre two spheres whose radii are $\sqrt{\left(\frac{I-A}{M}\right)}$ and $\sqrt{\left(\frac{I-C}{M}\right)}$, they must obviously pass all without the former and within the latter.

The greater the value of I, the narrower therefore will be the limits, for the rectangle under the sum and difference of the radii of the spheres is constant; so that the greater their sum, the less their difference.

In the particular class of bodies for which the three central principal moments are equal, they will be still more restricted, for then obviously must each system be equidistant from, and therefore envelope a sphere round, the centre of gravity; the different spheres for different values of the common moment being all concentric, that point being their common centre.

In such bodies, therefore, the different systems of equimomental axes in every plane assumed at pleasure will envelope a system of concentric circles round the projection on each plane of the centre of gravity: this is also evident from (10), where it appeared that in *every* body there exist two systems of parallel planes which possess the same property, viz. those parallel to the cyclic planes of the momental ellipsoid at the centre of gravity.

The most interesting class of bodies, with respect to the arrangement of their equimomental axes, are those for which two of the three principal central moments are equal, the third not having the same value. For in these, the momental ellipsoid at the centre of gravity, and therefore the ellipsoid of inertia its reciprocal to the concentric sphere $r^2 = 1$, will be of revolution, and so will also the whole central system of equimomental cones, the axis of revolution common to all these surfaces being that of the unequal moment.

Now, for every plane passing through that axis every system of equimomental axes (10) will envelope a conic confocal with the meridian of the ellipsoid of inertia, and which will be an ellipse or hyperbola, according as the common moment exceeds the greater of the two principal central moments, or is intermediate to the greater and less, the

asymptotes of the hyperbola in the latter case being the particular pair of axes which pass through the centre.

Hence every system of equimomental axes which pass all through the axis of unequal moment will envelope a central surface of the second order of revolution round that axis, and which will be an ellipsoid or hyperboloid, as the case may be. And, for different values of the common moment, the whole system of envelopes will be confocal with the ellipsoid of inertia at the centre of gravity, and the central system of equimomental cones will be obviously the asymptotic cones of the system of hyperboloids.

(It hence immediately appears, that if upon any plane assumed at will we project orthographically this system of confocal surfaces, then will the confocal system of conics, contours of the several projections, be the system of envelopes of equimomental systems of axes in that plane (10). But this, as well as the theorem itself, are particular cases of one still more general, which shall be subsequently noticed).

Of bodies of this class there are two varieties (quite distinct in many of their properties), according as the unequal moment is greater or less than those that are equal: in the former case the system of confocal ellipsoids will be all *prolate* spheroids, and the hyperboloids will be all of *two sheets;* in the latter the ellipsoids will be all *oblate* spheroids and the hyperboloids all of *one sheet:* this is obvious, for the system of confocal conics which have been supposed to generate these surfaces, in the former case revolve round their primary axis, and in the latter, round their secondary.

The rectilinear generatrices of each hyperboloid, in the latter case, are also a system of equimomental axes: this is evident, for they are in pairs, parallel to, and equidistant from, the sides of their asymptotic cones, which for each hyperboloid are a system of equimomental axes through the centre of gravity.

Bodies of the former variety possess the important property, that in them there exist two points for which *all* axes are principal, (5). These points are the two foci of the central spheroid of inertia and of its prolate system of confocal surfaces; that is, of the ellipsoid of revolution reciprocal to the momental spheroid at the centre of gravity.

For *all* axes through either of these foci pass through the axis of unequal moment, and are tangents to the infinitely slender surfaces of revolution which, terminated by the two foci, form the transition state between the ellipsoids and the hyperboloids of the confocal system: these axes are therefore *all* equimomental, and therefore (5) all principal.

Bodies of the second variety admit of no point for which *all* axes are principal, but all points on the focal circle of their oblate system of confocal surfaces of revolution admit of an *infinite number* of principal axes (5), which, for each point, lie all in the plane containing that point and the axis of revolution, that is, in the normal plane to the circle at the same point.

For all axes through any of these points which lie in the normal plane to the circle intersect the axis of unequal moment, and are tangents to the infinitely flat surfaces of revolution which, bounded by that circle, form the transition state between the ellipsoids and the hyperboloids: these axes are therefore all equimomental, and, as lying in the equators of the momental ellipsoids at each point, are therefore all principal.

For both varieties of body, that is, for all bodies of the class we are now considering, the axis of revolution is obviously principal for all points along it, which points also admit all of an infinite number of principal axes in planes perpendicular to that axis: but these, being particular cases of a more general property common to all bodies, which will be fully discussed in a subsequent article, we shall notice no further at present.

(*To be continued.*)

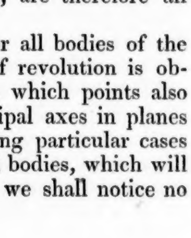
Trinity College, Dublin, May 1846.

MISCELLANEOUS NOTES ON DESCRIPTIVE GEOMETRY. NO. I.

By T. S. Davies, F.R.S.L. & E., F.S.A.

The following constructions are, I think, simpler than any which have been given by the French writers.

Prop. I. *To construct the perpendicular from a given point to a given plane.*



Let P_1PP_2 be the given plane exhibited by means of its traces, and a_1, a_2 the projections of the given point.

Through a_1 draw the perpendicular pP_1 to PP_1, and from p draw pP_2 perpendicular to the axis Pp. Make pp_2 at right angles to pP_1 and equal to pP_2, and join p_2P_1.

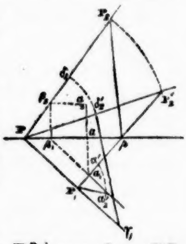
Again, draw a_1a_2' perpendicular to pP_1 and equal to aa_2; and from a_2' draw $a_2'a_2'$ perpendicular to P_1p_2. Then $a_2'a_2'$ is the perpendicular required.

Also, for the projections of the point of intersection, draw $a_2'a_1$ perpendicular to pP_1, and a_1a_2 perpendicular Pp, taking aa_2 equal to a_1a_2'. Then a_1, a_2 are the projections of the point in which the perpendicular meets the plane.

It is clear that the construction might have been similarly made on the vertical plane.

The truth of this construction will be so obvious to those who are familiar with the subject, as to render a formal demonstration superfluous.

Prop. II. *A point is situated in a given plane, and one of its projections is given to construct the* rabattement *of the plane and of the point in it, upon either of the coordinate planes.*



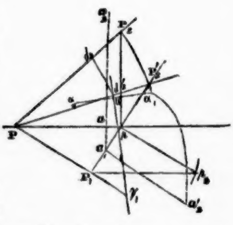
Find the vertical projection a_2 of the given point in the usual manner as indicated by the dotted lines.

Through a_1 draw the perpendicular P_1P_2', and make PP_2' equal to PP_2: then PP_2' is the *rabattement* of PP_2. In $a_1\beta_1$ make a_1a_2' equal to aa_2, and then P_1a' equal to P_1a_2'. Then a' is the rabattement of the point.

If a line in the plane P_1PP_2 is to be *rabatted*, let its traces be γ_1, δ_2. Make $P\delta_2'$ equal to $P\delta_2$, and join $\gamma_1\delta_2'$. This is the position of the line required.

The angle P_1PP_2' is that formed by the traces in their natural position: for finding which *directly*, this construction is well adapted.

Prop. III. *Given the projections of a point upon two coordinate planes, to find its projections upon any other two, both systems being rectangular.*



Let $\gamma_1\delta_2$ be the intersection of the new coordinate planes referred to the original ones whose axis is Pp; and let P_1PP_2 be one of the new coordinate planes. Also, let a_1a_2 be the given point referred to the original system.

1. Construct the *rabattement* of the plane P_1PP_2 about PP_1, (by Prop. II.)

2. Find the length a_2a' of the perpendicular from a_1a_2 to the plane P_1PP_2, (by Prop. I.)

3. Find the *rabattement* $\gamma_1\delta_2'$ of the line $\gamma_1\delta_2$, and the rabattement a_1 of the point a_1a_2, (by Prop. II.)

4. Draw a_1a_2 perpendicular to $\gamma_1\delta_2'$, so that aa_2 shall be equal to $a_2'a'$.

Then a_1a_2 is the point referred to the new system.

Scholium.

The use of oblique coordinate planes has not been introduced into Descriptive Geometry, except in one or two simple cases. No general system of construction by means of them has ever been even suggested; nor indeed does there seem to exist an impression that their introduction would be attended with any advantage in actual construction.

There is no doubt that the problems of the most usual practical occurrence do, *in general*, admit of more elegant construction by means of rectangular coordinate planes than by means of oblique: still my own experience convinces me that this rule is very far from being universal. At any rate, as a matter of mathematical interest, it seems advisable to give a development of such a system of construction, in order that we might have a choice of means (analogous to those we possess in the ordinary geometry of coordinates) of selecting the methods best adapted to any given problem.

Such a system, with the Editor's permission, I propose to give hereafter through the medium of this Journal.

College for Civil Engineers, Putney, March 20, 1846.

ANALYTICAL INVESTIGATIONS OF TWO OF DR. STEWART'S GENERAL THEOREMS.

By T. S. Davies, F.R.S.L. & E., F.S.A.

About two years ago I sent the Royal Society of Edinburgh "An Analytical Investigation of Dr. Matthew Stewart's General Theorems," which paper was printed in the *Transactions* of the Society. Those mathematicians who have looked into these theorems do not need to be informed that not the slightest clue to a general mode of investigation can be deduced from the solutions which Dr. Stewart himself gave of the first eight of them; or, at any rate, that no one has ever succeeded, by following his steps or imitating his processes, in proving the truth of the remoter and most general ones.

There are two of those theorems (the *seventh* and *eighth*) which Dr. Stewart demonstrated by geometrical consider-

ations, of which it did not fall in my plan in drawing up that paper to take special notice. I had, however, subjected these to the same process which I had found to be effective in treating those which had a certain analogy (though not a very close one) to the theorems which had been merely enunciated; and possibly the solutions so obtained might not be without interest to those geometers who have devoted any attention to these subjects.

I have given the steps with adequate detail to render the reading of the solutions more facile than they would be if much abbreviated: and I would remark that by the use of rectangular coordinates the expressions may be rendered more brief to the *eye*, although it does not appear likely that the reasoning itself could be materially abbreviated by such means.

PROP. VII. GENERAL THEOREMS.

"*Let there be any circle whose centre is* A, *and let* BCD *be a segment of the circle, and* BD *the chord of the segment; about the segment let there be any equilateral figure circumscribed touching the circle in the points* E, F, G, *etc., and let the two sides of the figure next* BD *meet* BD *in* H, K; *bisect the segment* BCD *in* F, *and join* AF; *in* AF *take the point* L

on the same side of the centre A *with the point* F, *and let the sum of the sides of the figure circumscribed about the segment into* HK *as the semidiameter to* AL; *draw* ML *perpendicular to* AL *meeting the circle in* M. *If from the points* E, F, G, *etc., the points of contact of the circumscribed figure, and the point* L, *there be drawn right lines to any point* N, *the sum of the squares of* EN, FN, GN, *etc., will be equal to the multiple of the sum of the squares of* LM, LN *by the number of the sides of the figure.*"

Dr. Stewart divides his demonstration into two cases, according as N is in the circumference of the given circle or not, the former being a "case of case" to the latter. In the method here employed it will be more convenient to consider the two cases dependent on the number of sides of the polygon being even or odd.

1. *The polygon having* $2n$ *sides.*

Let the given segment be $\frac{2n\pi}{m}$, and ρ the radius of the given circle. Take the line through the centre of the circle, viz. AF, as angular origin, which will also pass through an

angular point of the figure. Then it will be obvious from the figure itself that

$$\text{perimeter} = 4n\rho \tan \frac{\pi}{2m},$$

$$HK = 2\rho \sec \frac{\pi}{2m} \sin \frac{n\pi}{m},$$

$$AL = \frac{\rho \sin \frac{n\pi}{m}}{2n \sin \frac{\pi}{2m}},$$

and $$LM^2 = \rho^2 \left(1 - \frac{\sin^2 \frac{n\pi}{m}}{4n^2 \sin^2 \frac{\pi}{2m}}\right) \dots\dots\dots\dots (1).$$

Now the coordinates of the points of contact referred to the centre A and axis AF will be

$$\left(\rho, \frac{\pi}{2m}\right), \left(\rho, \frac{3\pi}{2m}\right) \dots \left\{\rho, \frac{(2n-1)\pi}{2m}\right\};$$

and if $r\theta$ denote the arbitrary point N, and the expressions for the several squares enunciated be formed, we shall have

$$NL^2 = r^2 - 2r.AL \cos\theta + AL^2$$

$$= r^2 - 2r\rho . \frac{\sin \frac{n\pi}{m} \cos\theta}{2n \sin \frac{\pi}{2m}} + \rho^2 . \frac{\sin^2 \frac{n\pi}{m}}{4n^2 \sin^2 \frac{\pi}{2m}} \dots (2),$$

and from (1, 2), we get at once

$$NL^2 + LM^2 = r^2 - r\rho \frac{\sin \frac{n\pi}{m}}{n \sin \frac{\pi}{2m}} \cos\theta + \rho^2 \dots (3).$$

Also taking the squares of the lines drawn from N to the points of contact equidistant from the circular origin in pairs, we shall have them represented by

$$\left.\begin{array}{l} r^2 - 2r\rho \cos\left(\theta - \frac{\pi}{2m}\right) + \rho^2 \\ r^2 - 2r\rho \cos\left(\theta + \frac{\pi}{2m}\right) + \rho^2 \end{array}\right\},$$

$$\left.\begin{array}{l} r^2 - 2r\rho\cos\left(\theta - \dfrac{3\pi}{2m}\right) + \rho^2 \\ r^2 - 2r\rho\cos\left(\theta + \dfrac{3\pi}{2m}\right) + \rho^2 \end{array}\right\},$$

. . . .

$$\left.\begin{array}{l} r^2 - 2r\rho\cos\left\{\theta - \dfrac{(2n-1)\pi}{m}\right\} + \rho^2 \\ r^2 - 2r\rho\cos\left\{\theta + \dfrac{(2n-1)\pi}{m}\right\} + \rho^2 \end{array}\right\}.$$

Expanding the cosines and adding, and putting S^2 for the sum of the squares of the lines, we have

$$S^2 = 2n(\rho^2 + \gamma^2) - 4r\rho\cos\theta\left(\cos\frac{\pi}{2m} + \cos\frac{3\pi}{2m} + \ldots\right).$$

But $\cos\dfrac{\pi}{2m} + \cos\dfrac{3\pi}{2m} + \ldots + \cos\dfrac{(2n-1)\pi}{2m} = \dfrac{\sin\dfrac{n\pi}{m}}{2\sin\dfrac{\pi}{2m}}$,

and hence $\quad S^2 = 2(\rho^2 + r^2) - 2r\rho\dfrac{\sin\dfrac{n\pi}{m}}{\sin\dfrac{\pi}{2m}}\cos\theta \ldots\ldots (4).$

Wherefore, multiplying (3) by $2n$, we have (4), which shews that in this case the proposition is true; viz. that

$$S^2 = 2n(NL^2 + LM^2).$$

2. *Let the polygon have* $2n + 1$ *sides.*

Take origin and axis as before, the axis now passing through one of the points of contact of the figure. Then we shall have, the segment being denoted by $\dfrac{(2n+1)\pi}{m}$,

$$\text{perimeter} = 2(2n+1)\rho\tan\frac{\pi}{2m},$$

$$HK = 2\rho\sec\frac{\pi}{2m}\sin\frac{(2n+1)\pi}{2m},$$

$$AL = \frac{\rho\sin\dfrac{(2n+1)\pi}{2m}}{(2n+1)\sin\dfrac{\pi}{2m}},$$

whence $$ML^2 = \rho^2 \left\{1 - \frac{\sin^2 \dfrac{(2n+1)\pi}{2m}}{(2n+1)^2 \sin^2 \dfrac{\pi}{2m}}\right\} \ldots\ldots (1),$$

and $$NL^2 + LM^2 = r^2 - 2r\rho \frac{\sin \dfrac{(2n+1)\pi}{2m} \cos\theta}{(2n+1)\sin\dfrac{\pi}{2m}} + \rho^2 \ldots (2).$$

Again, for the lines drawn to the points of contact, that drawn to the circular origin being taken alone, and those to the points of contact equidistant from it in pairs, we shall have

$$r^2 - 2r\rho\cos\theta + \rho^2,$$

$$\left.\begin{array}{l} r^2 - 2r\rho\cos\left(\theta - \dfrac{\pi}{m}\right) + \rho^2 \\ r^2 - 2r\rho\cos\left(\theta + \dfrac{\pi}{m}\right) + \rho^2 \end{array}\right\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left.\begin{array}{l} r^2 - 2r\rho\cos\left(\theta - \dfrac{n\pi}{m}\right) + \rho^2 \\ r^2 - 2r\rho\cos\left(\theta + \dfrac{n\pi}{m}\right) + \rho^2 \end{array}\right\}.$$

Adding all these, except the first, we get the sum of all the squares except that one:

$$= 2n(r^2 + \rho^2) - 4r\rho\cos\theta \left\{\cos\frac{\pi}{m} + \cos\frac{2\pi}{m} + \ldots \cos\frac{n\pi}{m}\right\},$$

$$= 2n(r^2 + \rho^2) - 2r\rho\cos\theta \left\{\frac{\sin\dfrac{(2n+1)\pi}{2m}}{\sin\dfrac{\pi}{2m}} - 1\right\}.$$

Add the first line to this: then there results

$$S^2 = (2n+1)(r^2 + \rho^2) - 2r\rho \frac{\sin\dfrac{(2n+1)\pi}{2m}}{\sin\dfrac{\pi}{2m}} \cos\theta \ldots (3).$$

Multiply (2) by $2n + 1$, and we have the result equal to (3); which therefore also proves the theorem in this case.

PROP. VIII. GENERAL THEOREMS.

"*Let there be any circle whose centre is* A, *and let* BCD *be a semicircle, and* BD *the diameter of the circle; about the semicircle let there be any regular figure described, and let the sides of the figure next to* BD *meet* BD *in* E, F; *bisect the semicircle in* G *and join* AG; *and in* AG *take the point* H *on the same side of the centre* A *with the point* G, *and let* AG *be to* AH *as the sum of the sides of the figure to* EF; *and let the rectangle* HAK *be equal to the square of the semidiameter, and* HL *be equal to* AH: *if from any point* M *there be drawn* MN, MO, MP, *etc., perpendicular to the sides of the figure circumscribed about the semicircle, and likewise there be drawn* ML *to the point* L; *twice the sum of the squares of the perpendiculars* MN, MO, MP, *etc., will be equal to the multiple of the square of* ML *by the number of the sides of the figure together with the multiple of the rectangle* KLA *by the same number.*"

1. *Let the figure have* $2n$ *sides.*

Taking the line AG as angular origin, the angular axis will pass through an angular point of the circumscribed figure. Denote the radius by ρ: then the polar equations of the sides of the figure whose points of contact lie respectively above and below the circular origin will be (Hutton's *Course*, vol. II. p. 264, *twelfth edition*),

$$\rho = r\cos\left(\theta - \frac{\pi}{4n}\right),\quad \rho = r\cos\left(\theta - \frac{3\pi}{4n}\right),\ldots.$$

$$\rho = r\cos\left\{\theta - \frac{(2n-1)\pi}{4n}\right\},$$

$$\rho = r\cos\left(\theta + \frac{\pi}{4n}\right),\quad \rho = r\cos\left(\theta + \frac{3\pi}{4n}\right),\ldots.$$

$$\rho = r\cos\left\{\theta + \frac{(2n-1)\pi}{4n}\right\},$$

and the perpendiculars from the arbitrary point $r\theta\,(M)$ upon these will be (*ib.*)

$$\rho - r\cos\left(\theta - \frac{\pi}{4n}\right),\quad \rho - r\cos\left(\theta - \frac{3\pi}{4n}\right),\ldots.$$

$$\rho - r\cos\left\{\theta - \frac{(2n-1)\pi}{4n}\right\},$$

$$\rho - r\cos\left(\theta + \frac{\pi}{4n}\right),\quad \rho - r\cos\left(\theta + \frac{3\pi}{4n}\right),\ldots .$$

$$\rho - r\cos\left\{\theta + \frac{(2n-1)\pi}{4n}\right\}.$$

Taking the sums of the squares of these in pairs as they stand beneath each other, and expressing the results in multiple cosines, we get

$$2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{\pi}{4n} + r^2\cos 2\theta\cos\frac{\pi}{n},$$

$$2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{3\pi}{4n} + r^2\cos 2\theta\cos\frac{3\pi}{n},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{(2n-1)\pi}{4n} + r^2\cos 2\theta\cos\frac{(2n-1)\pi}{n}.$$

Now the sum of these being taken, the column in $\cos 2\theta$ vanishes, its coefficient being $\dfrac{\sin\pi\cos\pi}{\sin\frac{\pi}{n}} = 0$; and the column in $\cos\theta$ has for its coefficient the value $\dfrac{\sin\frac{\pi}{4}\cos\frac{\pi}{4}}{\sin\frac{\pi}{4n}} = \dfrac{1}{2\sin\frac{\pi}{4n}}$:

and hence twice the sum of the squares, $2S^2$, of the $2n$ perpendiculars is

$$2S^2 = 4n\rho^2 + 2nr^2 - 4r\rho\frac{\cos\theta}{\sin\frac{\pi}{4n}} \quad\ldots\ldots\ldots (1).$$

Again, from the obvious properties of the figure,

$$\text{perimeter} = 4n\rho\tan\frac{\pi}{4n},$$

$$AH = \frac{\rho}{2n\sin\frac{\pi}{4n}},$$

$$AK = 2n\rho\sin\frac{\pi}{4n},$$

$$AL = \frac{\rho}{n\sin\frac{\pi}{4n}},$$

$$KL = \rho \,.\, \frac{2n^2 \sin^2 \dfrac{\pi}{4n} - 1}{n \sin \dfrac{\pi}{4n}},$$

$$KL.LA = 2\rho^2 - \frac{\rho^2}{n^2 \sin^2 \dfrac{\pi}{4n}},$$

$$ML^2 = LA^2 - 2LA.AM \cos LAM + AM^2$$

$$= \frac{\rho^2}{n^2 \sin^2 \dfrac{\pi}{4n}} - \frac{2r\rho \cos \theta}{n \sin \dfrac{\pi}{4n}} + r^2;$$

and hence $ML^2 + KL.LA = 2\rho^2 + r^2 - \dfrac{2r\rho \cos \theta}{n \sin \dfrac{\pi}{4n}} \ldots . (2).$

Whence multiplying this by $2n$, we have the same result as in (1); and this identity of value is that enunciated in the proposition, viz.

$$2S^2 = 2n\,(ML^2 + KL.LA).$$

2. *Let the figure have* $2n + 1$ *sides.*

In this case the angular axis will pass through the middle point of contact; and the perpendiculars from $r\theta$ upon the sides of the figure, whose points of contact are respectively above and below the circular origin, will be (omitting that upon the side whose point of contact is the circular origin)

$$\rho - r\cos\left(\theta - \frac{\pi}{2n+1}\right), \quad \rho - r\cos\left(\theta - \frac{2\pi}{2n+1}\right) \ldots .$$

$$\rho - r\cos\left(\theta - \frac{n\pi}{2n+1}\right),$$

$$\rho - r\cos\left(\theta + \frac{\pi}{2n+1}\right), \quad \rho - r\cos\left(\theta + \frac{2\pi}{2n+1}\right) \ldots .$$

$$\rho - r\cos\left(\theta + \frac{n\pi}{2n+1}\right).$$

Twice the sum of the squares of these being taken, and the results expressed in multiple cosines, we get

$$2\left\{2\rho^2 + r^2 - 4r\rho \cos\theta \cos\frac{\pi}{2n+1} + r^2 \cos 2\theta \cos\frac{2\pi}{2n+1}\right\},$$

$$2\left\{2\rho^2 + r^3 - 4r\rho\cos\theta\cos\frac{2\pi}{2n+1} + r^3\cos 2\theta\cos\frac{4\pi}{2n+1}\right\},$$

....

$$2\left\{2\rho^2 + r^2 - 4r\rho\cos\theta\cos\frac{n\pi}{2n+1} + r^2\cos 2\theta\cos\frac{2n\pi}{2n+1}\right\},$$

the sum of which is

$$2n\,(2\rho^2 + r^2) - 4r\rho\cos\theta\left\{\frac{1}{\sin\dfrac{\pi}{2n+1}} - 1\right\} - r^2\cos 2\theta\,. \,.(1).$$

Also double the square of the omitted perpendicular is

$$2\rho^2 + r^2 - 4r\rho\cos\theta + r^2\cos 2\theta,$$

which, added to (1), gives

$$2S^2 = (2n+1)\,(2\rho^2 + r^2) - 4r\rho\,.\,\frac{\cos\theta}{\sin\dfrac{\pi}{2n+1}} \;\ldots\ldots(2).$$

Again, we have, as in the preceding case,

$$\text{perimeter} = 2\,(2n+1)\,\rho\tan\frac{\pi}{2\,(2n+1)},$$

$$AH = \frac{\rho}{(2n+1)\sin\dfrac{\pi}{2\,(2n+1)}},$$

$$AK = (2n+1)\,\rho\sin\frac{\pi}{2\,(2n+1)},$$

$$AL = \frac{\pi}{(2n+1)\sin\dfrac{2\rho}{2\,(2n+1)}},$$

$$KL = \rho\,.\,\frac{(2n+1)^2\sin^2\dfrac{\pi}{2\,(2n+1)} - 2}{(2n+1)\sin\dfrac{\pi}{2\,(2n+1)}},$$

$$KL.LA = 2\rho^2 - \frac{4\rho^2}{(2n+1)^2\sin^2\dfrac{\pi}{2\,(2n+1)}},$$

$$LM^2 = \frac{4\rho^2}{(2n+1)^2\sin^2\dfrac{\pi}{2\,(2n+1)}} - \frac{4r\rho\cos\theta}{(2n+1)\sin\dfrac{\pi}{2\,(2n+1)}} + r^2;$$

whence, adding, we get

$$KL.LA + LM^2 = 2\rho^2 + r^2 - \frac{4r\rho\cos\theta}{(2n+1)\sin\dfrac{\pi}{2(2n+1)}},$$

$$\text{or } (2n+1)(KL\,LA + LM^2) = (2n+1)(2\rho^2 + r^2) - \frac{4r\rho\cos\theta}{\sin\dfrac{\pi}{2(2n+1)}} \,..(3).$$

The identity of (1) and (3) proves the truth of the theorem, also, when the number of sides is odd.

Royal Military Academy, Woolwich, March 25, 1846.

ON ARBOGAST'S FORMULÆ OF EXPANSION.

By Augustus De Morgan,
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§ 1. *General Theory of Derivatives.*

The theory of Arbogast has received so little attention in this country, that no excuse is necessary for an attempt to exhibit its rules in a short and comparatively easy manner. Arbogast himself was more occupied in proving the ease with which his method could be applied to very complicated cases, than in illustrating the connexion of its principles with those of other parts of analysis.

The first attempt of which I know, to write on this subject in English, is contained in the posthumous work* of Mr. West, which is a very complete attempt, as far as series of one variable are concerned. The next is that which I made myself in my work on the Differential Calculus, (at which time I did not know of Mr. West's work): and I am not aware of any other. I think that many mathematicians are under the impression that Argobast's method belongs to the *combinatorial analysis* of Hindenburg and his followers. This, however, any one who carefully examines both will find is not the case.

When Arbogast developes $\phi(a + bx + cx^2 + \ldots)$, he presumes

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* 'Mathematical Treatises, containing, 1. the Theory of Analytical Functions .... By the Rev. John West .... edited .... from his MSS. .... by the late Sir John Leslie ....' Edinburgh, 1838. 8vo.—Mr. West died in Jamaica in 1817, aged 61.

that $a+bx+cx^2+\ldots$ is some particular case of $F(a+x)$: so that the development is

$$\phi Fa + (\phi F)'a \,.\, x + (\phi F)''a \,.\, \frac{x^2}{2} + \ldots$$

Now $(\phi F)'a$ is $\phi' Fa \,.\, F'a$, or $\phi' a \,.\, b$, &c., whence he derives his fundamental rule, namely, that the coefficients of x, x^2, &c. are obtained from ϕa by successive differentiations (followed by divisions by 1, 2, 3, &c.) of ϕa with respect to an imaginary variable; on the supposition that a, b, c, &c. have the differential coefficients b, $2c$, $3d$, &c. From this method, partly by inspection, he obtains the subordinate rules which distinguish his treatise from others on the same subject. This is enough to shew that Arbogast's method begins by the use of the usual methods of analysis.

I shall first explain the general form which contains these rules, and then proceed to establish them, and to connect them with a problem of combinations, from which such simplifications as they admit of may be easily deduced.

There is a difficulty about notation, particularly in the case of series of two variables. Some will think it best to proceed by letters, a, b, c, e, &c.; others by superfixes or suffixes, as a, a', a'', &c., a_0, a_1, a_2, &c. Arbogast proceeds by letters with respect to x, and by superfixed accents with respect to y. I shall (at least where two variables are concerned) use both methods, denoting a series of one variable by

$$a + b'x + c''x^2 + e'''x^3 + f^{\mathrm{IV}}x^4 + \ldots,$$

and one of two variables by

$$\left.\begin{array}{llll} a + b'x & + c''x^2 & + e'''x^3 & \\ + b_{,}y & + c_{,}'xy & + e_{,}''x^2y & \\ & + c_{,,}y^2 & + e_{,,}'xy^2 & \\ & & + e_{,,,}y^3 & \end{array}\right\} + \text{\&c.}$$

For this I have a twofold reason. First, those who prefer either plan alone may drop that portion of my distinctions which is to them superfluous. Secondly, I have myself found this double system of notation a very useful check in the algebraical operations which this subject contains.

I now proceed to the definitions and rules of the system.

Let there be a function of any number of letters, say a, b, c, e, &c.; and let there be a number of *convertible and distributive* operations, α, β, γ, ϵ, &c., of which each acts only on one letter, α on a, β on b, &c., so that $\gamma\phi(a, b)$, for instance, is $= 0$.

Derivation, the process by which so much of algebraical development is performed, is the performance of the operations $\alpha\beta^{-1}$, $\beta\gamma^{-1}$, $\gamma\epsilon^{-1}$, &c. These give *partial derivatives,* and the complete derivative is obtained from the performance of $\alpha\beta^{-1} + \beta\gamma^{-1} + \gamma\epsilon^{-1} + \ldots$; *but only on this condition, that no term which is produced by any one of the partial derivatives shall be allowed to appear again as a result of any other.* So that, in fact, complete derivation means—the result of $\alpha\beta^{-1}$ + so much of that of $\beta\gamma^{-1}$ as is not given by $\alpha\beta^{-1}$ + so much of that of $\gamma\epsilon^{-1}$ as is not given by $\alpha\beta^{-1} + \beta\gamma^{-1}$ + &c. It is obvious that the order of the terms is convertible in the expression of this condition.

If the quantity operated upon be a function only of a, then it is useless to perform β or any subsequent operation except upon terms which have undergone the inverse operation. The inverse operation must produce its letter, or the process would be unmeaning: thus, if $\gamma^{-1}\phi(a, b)$ were not a function of c, $\gamma\gamma^{-1}\phi(a, b)$, which should be $\phi(a, b)$, would be 0. Thus again, beginning with a function of b only, the first derivative has the operation $\beta\gamma^{-1}$ only; the second has only $(\beta\gamma^{-1} + \gamma\epsilon^{-1})\beta\gamma^{-1}$, or $\beta^2\gamma^{-2} + \beta\epsilon^{-1}$. The third has only $\beta^3\gamma^{-3} + \beta^2\gamma^{-1}\epsilon^{-1}$ + ($\beta^2\gamma^{-1}\epsilon^{-1}$ rejected as having already occurred) + $\beta\zeta^{-1}$, and so on.

It is clear that any derivative formed from a function of b, by means of $\beta\gamma^{-1} + \ldots$, contains direct forms of β only, and inverse ones of all the others. The reason is that, (λ, μ, ν being consecutive operations) $\lambda\mu^{-1}$ must have been performed in any term, to introduce m, before anything is produced on which $\mu\nu^{-1}$ is effective: and μ is destroyed by μ^{-1}. If we write down a few of the above derivations from a function of b, we have, calling D the symbol of derivation,

$$\begin{aligned}
D &= \beta\gamma^{-1},\\
D^2 &= \beta\epsilon^{-1} + \beta^2\gamma^{-2},\\
D^3 &= \beta\zeta^{-1} + \beta^2\gamma^{-1}\epsilon^{-1} + \beta^3\gamma^{-3},\\
D^4 &= \beta\eta^{-1} + \beta^2\gamma^{-1}\zeta^{-1} + \beta^2\epsilon^{-2} + \beta^3\gamma^{-2}\epsilon^{-1} + \beta^4\gamma^{-4},
\end{aligned}$$

and so on. Here, by reason of the function containing only b, no term can contain any one letter, as e, except those in which the inverse operation of ϵ occurs to introduce it.

And it is plain from the mode of formation, that any direct operation, as ϵ, performed on any one derivative, gives the same result as the next, ζ, performed on the next derivative. For if any term of the first be $\beta^m\gamma^{-n}\epsilon^{-p}\zeta^{-q}\ldots$, which can come only once, there will come in the next the term

$\beta^m\gamma^{-n}\epsilon^{-p+1}\zeta^{-q-1}\ldots$; and ζ performed on the second yields the same as ϵ performed on the first. This, and the vanishing of all terms in the two, when ϵ and ζ are performed on terms which do not contain ϵ^{-1} and ζ^{-1}, proves the theorem alleged. This theorem is the immediate cause of the sufficiency of Arbogast's derivatives for their purpose. Its converse is as easily shewn, namely, that no other functions except these derivatives can have this property.

The formation of the derivatives in this very simple case, namely, in which the function operated upon is only a function of the letter belonging to the first operation, may be facilitated by the following method of selecting the terms to be retained. Operate only on the last operation of each term, except when the last but one in the term is also the next before the last in the series; in which case operate also upon the last but one. Thus, λ, μ, ν, ξ being consecutive, the term $\ldots\lambda^{-l}\nu^{-n}$ yields only $\ldots\lambda^{-l}\nu^{-n+1}\xi^{-1}$ to the next derivative, but $\ldots\lambda^{-l}\mu^{-m}\nu^{-n}$ yields

$$\ldots\lambda^{-l}\mu^{-m}\nu^{-n+1}\xi^{-1} + \ldots\lambda^{-l}\mu^{-m+1}\nu^{-n-1}.$$

This result is established by Arbogast upon a process of observation, but it admits of a very easy proof by combinations, as follows.

Let there be a box B, containing an unlimited number of counters, followed in succession by other boxes C, E, F, G, &c., all empty. Let it be allowable to remove a counter out of one box into the next *after* it: but let no other single operation be allowed. When n such operations have been performed, let the distribution in the boxes be called an n^{th} state. Let β, γ, ϵ, ζ, &c. be interpreted as directions to take a counter out of B, C, &c.; let γ^{-1}, ϵ^{-1}, ζ^{-1}, be interpreted as directions to put a counter into C, E, &c. Then it is clear that each step of transference is either $\beta\gamma^{-1}$, or $\gamma\epsilon^{-1}$, or $\epsilon\zeta^{-1}$, &c.; and also that what is written opposite to D above is the one possible first state, and opposite to D^2 we have all possible second states, and so on. For it is manifest that the way of deriving every $(n+1)^{\text{th}}$ state is to take every possible n^{th} state, throw a counter out of every box which has one into the next, and strike out every state which is thus brought about more than once, so often as it appears after the first time.

Now it is plain that the simplest way of converting the distribution $\infty\,0000$, &c. into $\infty\, p, q, r, s$, &c., is to take $p+q+r+$ &c. counters from B, and throw them into C at $p+q+r+$ &c. steps. Then take $q+r+$ &c. out of C and throw them into E, at $q+r+$ &c. steps. Then take $r+$ &c.

out of E and throw them into F, and so on. The same distribution may be gained by many permutations of the order of the steps, but this is enough. Now in the preceding process there is no one step but what may be described as follows: either one is taken out of the last occupied box, and thrown into the first which has till then been empty; or one is taken out of the last occupied box but one, and thrown into the last which is occupied. And since every $(n+1)^{\text{th}}$ state must be an n^{th} state with one step more, and there is no state of which the final process need be anything but that of the *last or last but one* just described, it follows that if this process be applied to *every* n^{th} state, it will give *every* $(n+1)^{\text{th}}$ state.

Again, if in every n^{th} state in which there are one or more counters in, say C, we throw one counter into E, we shall have every $(n+1)^{\text{th}}$ state in which there are one or more counters in E: and the same for E and F, &c. This proves the rule for the successive operations upon successive derivatives, otherwise established above.

All the β-operations included in $\beta^{\pm m}$ are convertible with derivations, when the function is one of b only. A look at the process will establish this; which is moreover no more than saying, that if m balls be added to or taken from the first box, it matters nothing whether this be done before or after the establishment of any state. Accordingly, the derivatives of $\beta^{-m}(\phi b)$ contain the following operations:

$$D^0 \text{ contains } \beta^{-m}, \qquad D \text{ contains } \beta^{-(m-1)}\gamma^{-1},$$
$$D^2 \quad \ldots\ldots \quad \beta^{-(m-1)}\epsilon^{-1} + \beta^{-(m-2)}\gamma^{-2},$$

and so on.

Let us now suppose that every such operation, as ϵ for instance, either introduces or takes away multipliers which are functions of its letter, as e; so that

$$\epsilon . P \text{ must mean } P \times \text{some function of } e.$$

The consequence is, that derivation is altogether a convertible operation, when the conversion is made with letters which the rule of the last or last but one renders inoperative. Thus, though $D(\beta\gamma^{-1})$ is not $\beta D\gamma^{-1}$ or $\beta\epsilon^{-1}$, but $\beta^2\gamma^{-2} + \beta\epsilon^{-1}$, yet $D\beta\gamma^{-1}\epsilon^{-1}$ is the same thing as $\beta D(\gamma^{-1}\epsilon^{-1})$. And thus we see that $D\{\Sigma\beta^m\gamma^{-n}\epsilon^{-p}..\}$ is made up of $\Sigma\{\beta^m D.\gamma^{-n}\epsilon^{-p}..\}$ together with a term from every term in which β comes last but one and γ last.

From this we easily prove the following theorem, for any function of any letter, the first one, a, for example,

$$D^m = aD^{m-1}\beta^{-1} + a^2 D^{m-2}\beta^{-2} + \ldots + a^m\beta^{-m}.$$

Let this be true for any one case; say that

$$D^{m-1} = aD^{m-2}\beta^{-1} + a^2D^{m-3}\beta^{-2} + \ldots + a^{m-2}D\beta^{-(m-2)} + a^{m-1}\beta^{-(m-1)}.$$

Perform D on both sides,

$$D^m = D(aD^{m-2}\beta^{-1}) + D(a^2D^{m-3}\beta^{-2}) + \ldots.$$

Now in every term except the last, a cannot in any case be the last letter but one, and in the last term

$$D(a^{m-1}\beta^{-(m-1)}) \text{ yields } a^{m-1}D\beta^{-(m-1)} + a^m\beta^{-m}.$$

Hence, inverting the order of D and a in all but the last, and substituting for the last as just found, we see that the theorem is true for D^m, if true for D^{m-1}; but

$$D = a\beta^{-1}, \quad D^2 = aD\beta^{-1} + a^2\beta^{-2},$$

so that it is true in all cases.

In the problem of combinations, this theorem is the immediate consequence of the following. If we wish to ascertain all the m^{th} states which can arise from an unlimited number of counters in A, none in B, C, &c., we must collect all the $(m-1)^{\text{th}}$ states which arise from 1, 0, 0, 0, &c. in B, C, &c., all the $(m-2)^{\text{th}}$ states from 2, 0, 0, 0, in B, C, &c., all the $(m-3)^{\text{th}}$ states from 3, 0, 0, 0 in B, C, &c., and so on, ending with the state of m, 0, 0, &c. in B, C, &c. And it is evident that the law of rejection is here inoperative: it is impossible that any state consequent upon p, 0, 0, &c. and q, 0, 0, &c. can be identical, if p and q be different numbers.

§. 2. *Application to Functions of one Variable.*

For use, undoubtedly development by means of derivation is an *application* of the differential calculus: but in theory it is an *extension*. And this, although the last operations must be those of the differential calculus: just as the last operations of the differential calculus itself must be those of algebra.

There is nothing that so much tends to destroy a proper perception of the full extent of a method, and of its separate meaning, as previous knowledge of equivalent processes.

In the function $\phi(a + bx + cx^2 + ex^3 + \ldots)$ which we might expand by Maclaurin's theorem, let us ask how we can avoid the change of x into $x + \xi$, by processes performed upon a, b, c, &c. The answer is, that for a we must write $a + b\xi + c\xi^2 + e\xi^3 + \text{\&c.}$; for b we must write $b + 2c\xi + 3c\xi^2 + \ldots$; for c we must write $c + 3e\xi + 6f\xi^2 + 10g\xi^3 + \ldots$; for e we must write $e + 4f\xi + 10g\xi^2 + 20h\xi^3 + \ldots$; and so on. The function

ϕ, thus altered, being ϕ_1, the limit of $\phi_1 - \phi$ divided by ξ is one set of *derivative coefficients*, complete. The partial *derivatives* are got by making the single letters vary as above.

It is obvious that the complete derivative amounts to the result obtained by differentiating with respect to a, b, c, &c., considered as functions of some one imaginary variable, and then writing b, $2c$, $3e$, &c. as the differential coefficients of a, b, c, &c. This was Arbogast's first process, and by itself is a great saving of trouble. It is the process for the development of $\phi\,\{a + b\,(x + \xi) + c\,(x + \xi)^2 + ..\}$ and by making $x = 0$, *before the operations*, or working upon ϕa only, gives the process required in the development of $\phi\,(a + bx + cx^2 + ..)$.

I have called the preceding a derivation because Arbogast did so. But this derivation is *differentiation*, and nothing more. He soon drops it for his *divided derivation*, which is the real distinctive feature of his system, and which consists in *differentiation accompanied by integration*, to which, and to which only, I shall in future apply the term derivation.

If we look at $\phi\,(a + b)$, we see in the development the following theorem. The operation which furnishes $\phi\,(a + b)$ is of the form

$$1 + \alpha\beta^{-1} + \alpha^2\beta^{-2} + \alpha^3\beta^{-3} + \ldots,$$

giving, if we return to the problem of combinations, every first, second, third, &c. state of which ∞ 0 0 0, &c. is susceptible, on the condition that no advance shall be made beyond the second box. Here α means differentiation with respect to a, and β integration (from $b = 0$) with respect to b. The change of a into $a + c$ in $\phi(a+b)$ is equivalent to the addition of $\alpha\gamma^{-1} + \alpha\gamma^{-2} + ..$, and further to the change of α into $\alpha + \alpha\gamma^{-1} + \alpha\gamma^{-2} + ..$ of α^2 into $\alpha^2 + \alpha^2\gamma^{-1} + \alpha^2\gamma^{-2} + ..$ &c. on the second side.

Suppose P_n denotes the sum of terms like the preceding, answering to every possible final state (the initial state included) of which the initial state is ∞ 0 0 0 0, &c., the n^{th} being the last box which has ever received a counter. It is plain that P_{n+1} can be formed from P_n by combining with every case in P_n, separately, 0 in the $n + 1^{\text{th}}$ box, 1 in the $(n + 1)^{\text{th}}$ box, 2 in the same, &c., which, as all the counters must be originally brought from the first box, gives the performance of

$$1 + \alpha\nu^{-1} + \alpha^2\nu^{-2} + ..$$

upon every term of P_n, ν denoting the operation which comes n^{th} after α. Now this is precisely the process by which $\phi(a_0 + a_1 + .. + a_{n-1})$ is changed into $\phi\,(a_0 + .. + a_n)$, by writing $a_0 + a_n$ for a_0. So that the development of $\phi(a + b + c + e + ..)$

is the result of the sum of the operations arising from every possible final state in which as many of the successive boxes are or have been occupied as there are letters in a, b, c, e, &c. Whence $\phi(a+b+c+..) = \phi a + D\phi a + D^2\phi a + \ldots$, D being the operation described in the first section, on the supposition that α, β, γ, &c. mean differentiations with respect to a, b, c, &c., and that β^{-1}, γ^{-1}, &c. mean the corresponding $\int_0$ integrations.

If for b we write bx, for c, cx^2, and so on, the dimensions easily shew that

$$\phi(a+bx+cx^2+..) = \phi a + D\phi a.x + D^2\phi a.x^2 + ..;$$

but this will be better seen as follows. The general equation $\phi(a+b+..) = \phi a + D\phi a + ..$ obliges us to interpret $a+b+..$ as $a + Da + ..$, whence b, c, e, &c. are the successive derivatives of a: as, indeed, appears from the rule. This derivation preserving its notation, let the operations of the original theorem, $\beta^{\text{v}}, \gamma^{\text{v}}$, &c. include multiplication by x, x^2, &c.: the development of the series of powers of x will easily follow.

It is hardly necessary to shew that the special definitions of α, β, &c. above given satisfy the conditions. Indeed, so restrictive are the conditions, that a person accustomed to the calculus of operations will wonder what, except differential coefficients, they could mean. But it is true that other meanings might be given: the difficulty would lie in finding operations in which those meanings could be made useful.

By the rule of derivation established (not attending to the restriction of the last or the last but one), we have

$$D\phi a = \phi' a.b,$$

$$D^2\phi a = \phi' a.c + \phi'' a\,\frac{b^2}{2},$$

$$D^3\phi a = \phi' a.e + \phi'' a.bc + \begin{pmatrix}\phi'' a.b.c\\ \text{rejected}\end{pmatrix} + \phi''' a\,\frac{b^3}{2.3},$$

and so on. But the rule of the last or last but one, or a theorem already proved to which it led, shews that

$$D^n\phi a = \phi' a D^{n-1}b + \phi'' a D^{n-2}\,\frac{b^2}{2} + \phi''' a D^{n-3}\,\frac{b^3}{2.3} + .. + \phi^{(n)} a\,\frac{b^n}{2.3..n},$$

more conveniently written

$$D^n\phi a = \phi' a D^{n-1}b + \frac{\phi'' a}{2} D^{n-2}b^2 + \frac{\phi''' a}{2.3} D^{n-3}b^3 + .. + \frac{\phi^{(n)} a}{2.3..n}\, b^n.$$

The values of $D^m b^n$ are formed with very little practice almost as fast as they can be written. The only thing worth men-

tioning is the convenience of making the division which is to arise from the integration before the multiplication, or in abatement of the multiplication, which arises from the differentiation. Thus the derivatives of b^7 are

$$7b^6c,\ 7b^6e + 21b^5c^2,$$
$$7b^6f + 42b^5ce + 35b^4c^3,$$
$$7b^6g + 42b^5cf + 21b^5e^2 + 105b^4c^2e + 35b^3c^4,$$

and so on.

That fractional coefficients cannot enter, may be inferred from their general form, which may be obtained as follows.

If the initial state of the boxes B, C, E, F, &c. be n, 0, 0, 0, &c., the m^{th} derivative of b^n contains a term arising from every m^{th} state which can be thence produced. Now it is evident that in every removal from one box to the next, the process gives a multiplier, the number in the losing box before the loss, and a divisor, the number in the gaining box after the gain. Consequently the coefficient depends only on the numbers in the several boxes after the final step: for each divisor which comes in at any accession afterwards to be removed, is compensated by the multiplier introduced at the removal. By the time then that n, 0, 0, &c. has become p, q, r, &c the only effective multipliers are $n, n-1, \ldots n-p+1$, and the divisors are $1, 2, 3..q$, $1, 2, 3..r$, &c. Hence the coefficient which multiplies $b^pc^qe^r$, &c. in any derivative of b^n, is evidently

$$\frac{1.2.3..n}{1.2.3..p \times 1.2.3..q \times 1.2.3..r \times \&c.}.$$

This form also shews how it happens that the coefficients must be the same in $b^qc^re^r..$ or in $b^rc^qe^p..$ as in $b^pc^qe^r..$

With respect to the power of this method, none can judge but those who have tried both it and the substitutes for it. There is no producing conviction of the superiority of any process by description. But if any one will write down for himself even as much as

$$(a + bx + cx^2)^5 = a^5 + Da^5.x + D^2a^5.x^2 + \&c.$$
$$= a^5 + 5a^4bx + (5a^4c + 10a^3b^2)\,x^2$$
$$+ (20a^3bc + 10a^2b^3)\,x^3 + (10a^3c^2 + 30a^2b^2c + 5ab^4)\,x^4$$
$$+ (30a^2bc^2 + 20ab^3c + b^5)\,x^5 + (10a^2c^3 + 30ab^2c^2 + 5b^4c)\,x^6$$
$$+ (20abc^3 + 10b^3c^2)\,x^7 + (5ac^4 + 10b^2c^3)\,x^8$$
$$+ 5bc^4x^9 + c^5x^{10},$$

forming the derivatives by the method of the last or last but one,

and remembering $e = 0$, $f = 0$, &c. whenever they arise, he may then try the same by the binomial theorem, and decide the question for himself. Nor is this by any means a case peculiarly favourable to the method. Take the reversion of the series $y = a_1 x + a_2 x^2 + . .$ into $x = A_1 y + A_2 y^2 + . .$ and determine A_1, A_2, &c. up to A_{20} *first* by Arbogast's method (which will be the safest plan), and then by any other, until he is satisfied as to the relative ease of the two methods; and this will be a proper trial, with justice to the former method.

The point of view in which the calculus of operations places the main result is remarkable. It is

$$\phi \left\{(1 - xD)^{-1} a\right\} = (1 - xD)^{-1} \phi a,$$

shewing the convertibility of the ordinary operation, ϕ, of algebra, with the operation $(1 - xD)^{-1}$; and we have

$$\frac{1}{1 - xD}\, \phi a = \varepsilon^{\left(\frac{xD}{1 - xD}\, a\right) \frac{d}{da}}\, \phi a.$$

But we must not use the calculus of operations in obtaining results. For that calculus assumes a definite and permanent subject of operation, as a: while derivation introduces new subjects of operation at every step.

§. 3. *Application to Functions of two Variables.*

We have now obtained the equation

$$\phi (a + Da.x + D^2 a.x^2 + ...) = \phi a + D\phi a.x + D^2 \phi a.x^2 + ...$$

in which we may, if we please, make the operation D include, besides its hitherto expressed meaning, multiplication by x, and thus write the preceding as $\phi(a + Da + ...) = \phi a + D\phi a + ...$ This I shall not do, but I shall write the symbol D_x instead of D, as a distinction from other derivations presently to be noticed, and I shall write a, b', c'', e''', &c. for a, b, c, e, &c., or a, Da, $D^2 a$, &c.

If we now take a derivation of another kind, denoted in full by yD_y, in which the D_y derivatives of a are $b_{,}$ $c_{,,}$ $e_{,,,}$ &c.; those of b' are $c'_{,}$ $e'_{,,}$ $f'_{,,,}$ &c.; those of c'' are $e''_{,}$ $f''_{,,}$ $g''_{,,,}$ &c.; we shall find that $1 + D_y.y + D^2_y.y^2 + ...$ performed upon

$$a + b'x + c''x^2 + ...$$

is

$$a + b'x + b_{,}y + c''x^2 + c'_{,}xy + c_{,,}y^2 + ...$$

and by a repetition of the above theorem, we find that the development of $\phi(a + b'x + b_{,}y + ...)$ is

$$\phi a + D_x\phi a.x + D_y\phi a.y + D_x^{\,2}\phi a.x^2 + D_y D_x \phi a.xy + D_y^{\,2}\phi a.y^2 + ...$$

in which the coefficient of $x^m y^n$ is $D_x^m D_y^n \phi a$. The convertibility of D_x and D_y is proved either from the nature of the operations, or from an inversion of the preceding process. And the convertibility of D_x or D_y with $\frac{d}{da}$ is proved by differentiating $\phi(a + Da.x + \ldots) = \phi a + \ldots$ with respect to a, and comparing the result with $\phi'(a + \ldots)$ obtained from the theorem.

The above rule is not difficult to use, though not the easiest which the subject admits of. Let it be required to exhibit the value of $D_y^2 D_x^2 \phi a$. We have

$$D_x^2 \phi a = \phi' a.c'' + \frac{\phi'' a}{2} b'^2,$$

$$D_y D_x^2 \phi a = \phi' a.e_,'' + \frac{\phi'' a}{2} . 2b_, c'' + \frac{\phi'' a}{2} 2b' c_,' + \frac{\phi''' a}{2.3} 3b_, b'^2,$$

$$D_y^2 D_x^2 \phi a = \phi' a.f_{,,}'' + \frac{\phi'' a}{2} 2b_, e_,'' + \frac{\phi'' a}{2} (2b' e_{,,}' + c_,'^2 + 2b_, e_,'' + 2c_{,,} c'')$$

$$+ \frac{\phi''' a}{2.3} (6b_, b' c_,' + 3b_,^2 c'') + \frac{\phi''' a}{2.3} (6b_, b' c_,' + 3c_{,,} b'^2) + \frac{\phi^{iv} a}{2.3.4} 6b_,^2 b'^2,$$

in which the terms occurring twice are to be rejected. It must be remembered that the rule of the last or last but one does not apply to any thing but derivatives of a function of one quantity only.

If we take $b'x + b_, y + \ldots$ as an increment of a, use Taylor's theorem, and also the preceding theorem applied to the powers of the increment, we have the following theorem:

$$D_x^m D_y^n \phi a = \phi' a D_x^m D_y^n a + \frac{\phi'' a}{2} D_x^m D_y^n a^2 + \ldots$$

$$+ \frac{\phi^{(m+n)} a}{2.3 \ldots m+n} D_x^m D_y^n a^{m+n},$$

in which a is made to vanish in the derivatives. Now it is obvious that $D_x^m D_y^n a^{m+n+k}$, k being a positive integer, must vanish with a, for no derivation lowers a by more than one dimension. Hence, we have the following,

$$D_x^m D_y^n \phi a = D_x^m D_y^n \{\phi(a + z) - \phi a\},$$

on condition that all the derivatives of a be transferred to z, and z be made finally to vanish.

We have now two modes of derivation, D_x and D_y, in which the symbols x and y might have implied that multiplication by x or by y is a part of the operation. Let

there be two systems of derivation in which multiplication by $\frac{x}{y}$ or by $\frac{y}{x}$ might have been a part of the operation, and let their symbols be $D_{x:y}$ and $D_{y:x}$: when we begin with D_x derivations, we shall always want to combine $D_{y:x}$ derivations with them; accordingly, in $D^m{}_{y:x}D_x{}^n\phi a$, $y:x$ is superfluous; the colon alone will do, or we may write $D_:{}^m D_x{}^n\phi a$. In $D_{y:x}$ derivation, let the derivatives be as follows:

$$\begin{array}{llllllll}
\text{of } a & 0 & 0 & 0 & 0 & 0 & \&\text{c.} \\
b' & b_{,} & 0 & 0 & 0 & 0 & \&\text{c.} \\
c'' & c_{,}' & c_{,,} & 0 & 0 & 0 & \&\text{c.} \\
e''' & e_{,}'' & e_{,,}' & e_{,,,} & 0 & 0 & \&\text{c.,}
\end{array}$$

and so on.

By interchanging superfixes and suffixes, we get the $D_{x:y}$ derivations of $a, b_{,}, c_{,,}, e_{,,,}$, &c. The separate letters a, b, c, e, &c. uniformly represent coefficients of terms of the same order. Thus f is the index of the fourth order, and must always appear with four accents, either as f^{iv}, $f_{,}'''$, $f_{,,}''$, $f_{,,,}'$, or f_{iv}. As I before observed, those who think this superfluous, may use only one letter; but I should recommend them to do as is here done.

Take $\phi(a)$ and perform the operations $1 + D_x . x + D_x^2 . x^2 + ..$ and $1 + D_: . x^{-1}y + D_:^2 . x^{-2}y^2 + ..$ both internally and externally. We have then

$$\phi(a + b'x + b_{,}y + ..) = \phi a + D_x\phi a . x + D_: D_x\phi a . y$$
$$+ D_x^2\phi a . x^2 + D_: D_x^2\phi a . xy + D_:^2 D_x^2\phi a . y^2$$
$$+ D_x^3\phi a . x^3 + D_: D_x^3\phi a . x^2y + D_:^2 D_x^3\phi a . xy^2 + D_:^3 D_x^3\phi a . y^3 + ..$$

At the same time it is obvious that if the beginning had been made with $\phi(a + b_{,}y + ..)$ followed by $D_{x:y}$ derivations, and if $D_:$, written before D_y, had signified $D_{x:y}$, we should have had

$$\phi(a + b'x + b_{,}y + ...) = \phi a + D_: D_y\phi a . x + D_y\phi a . y + ...$$

from which we obtain, by equating the coefficients of $x^m y^n$,

$$D^n{}_{y:x} D_x{}^{m+n}\phi a = D^m{}_{x:y} D_y{}^{m+n}\phi a,$$

for all positive values of m and n from zero inclusive. Thus we have

$$D^n{}_{y:x} D_x{}^n\phi a = D_y{}^n\phi a, \quad D^m{}_{x:y} D_y{}^m\phi a = D_x{}^m\phi a.$$

To look at these results by the method of combinations,

let us suppose the following series of boxes, of which A has an unlimited number of counters and the rest none,

$$\begin{array}{ccccccc}
 & & & E''' & & & \\
 & & C'' & & & & \\
 & B' & & E_{,}'' & & & \\
A & & C_{,}' & & & & \&c. \\
 & B_{,} & & E_{,,}' & & & \\
 & & C_{,,} & & & & \\
 & & & E_{,,,} & & &
\end{array}$$

In $D_x^m \phi a$ we have, as has been seen, a term descriptive of every m^{th} state made by taking counters from A into B', C'', &c. at m steps. Now, since there are but m steps out, there are but m steps back again: so that if we changed the direction of transference, there is no result from m back-steps, except finally bringing all the counters back into A, and leaving B', C'', &c. empty. But there are no more steps from B' to A than from B' to B; no more from C'' to A than from C'' to $C_{,,}$, and so on: hence if, after establishing any m^{th} state along A, B', C'', &c., we make m vertical transferences, we must end by transposing all that were in B into $B_{,}$, all there were in C'' into $C_{,,}$, and so on, in neither more nor less than m steps. Hence, from the connexion previously established between the m^{th} states and m^{th} derivations, we have $D_{:}^m D_x^m \phi a = D_y^m \phi a$. And if we were, having established an m^{th} state in A, B', C'', &c., to make only $m-1$ vertical steps downwards, it follows that we produce neither more nor less than all the states which would have been obtained by establishing a corresponding m^{th} state in A, $B_{,}$, $C_{,,}$, &c. and making one vertical step upwards; and so on.

This establishes

$$D_{y:x}^{m-1} D_x^m \phi a = D_{x:y} D_y^m \phi a,$$

and similarly for the rest.

The reader will perhaps have noticed, that $D_{y:x}$ derivation is done by the operation $D_y D_x^{-1}$, for the passage from C'' to $C_{,}'$, for instance, may be made by $C'' E_{,}'' C_{,}'$, or by $C'' B' C_{,}'$. If we write $D_y D_x^{-1}$ for $D_{y:x}$, then $D_{y:x}^n D_x^{m+n}$ becomes $D_y^n D_x^n$, and the two modes of developing $\phi(a + b'x + b_{,}y + \&c.)$ are identical. But it is to be remembered that, without an extension, we cannot affirm the convertibility of D_x^m and D_y^n for all values, positive and negative, of the exponents of operation. If indeed we complete our system thus,

$$
\begin{array}{cccccccc}
a_{-\prime\prime\prime}''' & & c_{-\prime\prime}^{\text{iv}} & & f_{-\prime}^{\text{v}} & & h^{\text{vi}} & \\
 & b_{-\prime\prime}''' & & e_{-\prime}^{\text{iv}} & & g^{\text{v}} & & \\
a_{-\prime}'' & & c_{-\prime}''' & & f^{\text{iv}} & & h_{\prime}^{\text{v}} & \\
 & b_{-\prime}'' & & e''' & & g_{\prime}^{\text{iv}} & & \\
a_{-\prime}' & & c'' & & f_{\prime}''' & & h_{\prime\prime}^{\text{iv}} & \\
 & b' & & e_{\prime}'' & & g_{\prime\prime}''' & & \\
a & & c_{\prime}' & & f_{\prime\prime}'' & & h_{\prime\prime\prime}''' & \&c. \\
 & b_{\prime} & & e_{\prime\prime}' & & g_{\prime\prime\prime}'' & & \\
a_{\prime}^{-\prime} & & c_{\prime\prime} & & f_{\prime\prime\prime}' & & h_{\text{iv}}'' & \\
 & b_{\prime\prime}^{-\prime} & & e_{\prime\prime\prime} & & g_{\text{iv}}' & & \\
a_{\prime\prime}^{-\prime\prime} & & c_{\prime\prime\prime}^{-\prime} & & f_{\text{iv}} & & h_{\text{v}}' & \\
 & b_{\prime\prime\prime}^{-\prime\prime} & & e_{\text{iv}}^{-\prime} & & g^{\text{v}} & & \\
a_{\prime\prime\prime}^{-\prime\prime\prime} & & c_{\text{iv}}^{-\prime\prime} & & f_{\text{v}}^{-\prime} & & h_{\text{vi}} &
\end{array}
$$

with the understanding that all the negatively accented letters are made to vanish at the end of the process, we may then declare the four modes of derivation to be entirely convertible.

Leaving these extensions, however, we proceed to the details of development of $\phi(a + \&c.)$ by means of D_x derivations

and $D_{y:x}$ ones performed upon them. The coefficient of $x^m y^n$ is $D^n_{y:x} D_x^{m+n} \phi a$, or $D_{:}$ is to be performed n times upon

$$\phi' a \,.\, D^{m+n-1} b' + \frac{\phi'' a}{2} D^{m+n-2} b'^2 + \ldots + \frac{\phi^{(m+n)} a}{2 \ldots m+n} b^2 b'^{m+n}.$$

Now, because all the $D_{:}$ derivatives of a vanish, the process is wholly inoperative upon $\phi' a$, $\phi'' a$, &c.; and it is clear that by the difference of dimensions (derivation never producing change of dimension) no term of $D_{:}^p D_x^q b'^r$ can ever be identical with any sum of $D_{:}^s D_x^t b'^u$ if r and u be different. Hence the coefficient of $x^m y^n$ is

$$\phi' a D_{:}^n D_x^{m+n-1} b' + \frac{\phi'' a}{2} D_{:}^n D_x^{m+n-2} b'^2 + \ldots + \frac{\phi^{(m+n)}}{2 \ldots m+n} D_{:}^n b'^{m+n}.$$

It may be worth while to give an instance of the truth of the equation $D_{:}^m D_x^m b'^n = D_y^m b'^n$. Let us construct $D_{:}^3 D_x^3 b'^4$,

$$D_x b'^4 = 4b'^3 c'',$$

$$D_x^2 b'^4 = 4b'^3 e''' + 6b'^2 c''^2,$$

$$D_x^3 b'^4 = 4b'^3 f^{iv} + 12 b'^2 c'' e''' + 4b' c''^3.$$

We cannot now use the rule of the last or last but one, but must proceed with every letter, dropping terms already obtained. I begin from the last letters in each term:

$$\begin{aligned} D_{:} D_x^3 b'^4 = {} & 4b'^3 f_{,}''' + 12 b'^2 b_{,} f^{iv} + 12 b'^2 c'' e_{,}'' + 12 b'^2 c_{,}' e''' \\ & + 24 b' b_{,} c'' e''' + 12 b' c''^2 c_{,}' + 4 b_{,} c''^3 \end{aligned}$$

$$\begin{aligned} D_{:}^2 D_x^3 b'^4 = {} & 4b'^3 f_{,,}'' + 12 b'^2 b_{,} f_{,}''' + 12 b' b_{,}^2 f^{iv} + 12 b'^2 c'' e_{,,}' \\ & + 12 b'^2 c_{,}' e_{,}'' + 24 b' b_{,} c'' e_{,}'' + 12 b'^2 c_{,,} e''' \\ & + 24 b' b_{,} c_{,}' e''' + 12 b_{,}^2 c'' e''' + 12 b' c''^2 c_{,,} \\ & + 12 b' c'' c_{,}'^2 + 12 b_{,} c''^2 c_{,}' \end{aligned}$$

$$\begin{aligned} D_{:}^3 D_x^3 b'^4 = {} & 4b'^3 f'_{,,,} + 12 b'^2 b_{,} f_{,,}'' + 12 b' b_{,}^2 f_{,}''' + 4 b_{,}^3 f^{iv} \\ & + 12 b'^2 c'' e_{,,,} + 12 b'^2 c_{,}' e_{,,}' + 24 b' b_{,} c'' e_{,,}' \\ & + 12 b'^2 c_{,,} e_{,}'' + 24 b' b_{,} c_{,}' e_{,}'' + 12 b_{,}^2 c'' e_{,}'' + 24 b' b_{,} c_{,,} e''' \\ & + 12 b_{,}^2 c_{,}' e''' + 12 b_{,}^2 c'' e_{,}'' + 12 b_{,}^2 c_{,}' e''' + 24 b' c'' c_{,}' c_{,,} \\ & + 12 b_{,} c''^2 c_{,,} + 4 b' c_{,}'^3 + 12 b_{,} c'' c_{,}'^2. \end{aligned}$$

Now this term occurs in $D_{:}^3 D_x^7 \phi a$, the coefficient of $x^4 y^3$, in which it is the coefficient of $\phi^{iv} a \div 2.3.4$.

But $$D^3_{y:x} D_x^7 \phi a = D^4_{x:y} D_y^7 \phi a;$$

in which last the coefficient of $\phi^{iv} a \div 2.3.4$ is $D_{:}^4 D_y^3 b_{,}^4$, which is therefore $= D_{:}^3 D_x^3 b'^4$. And thus we have, generally,

$$D^m_{y \cdot x} D_x^n b'^p = D^{n+p-m}_{x:y} D_y^n b_{,}^p.$$

Now $D_y^3 b_{,}^4$ is only $D_x^3 b'^4$ with superfixes and suffixes interchanged; and $D_{:}^4 D_y^3 b_{,}^4$ will only be $D_{:}^4 D_x^3 b'^4$ with similar

interchanges. From this we gather, that if one more $D_{y:x}$ derivation be made in the preceding result, superfixes and suffixes will be interchanged, other things remaining the same. And this will be found on trial to be the case.

As another instance, we try $D_{:}^{4}D_{x}^{2}b'^{2}$, which ought to give $D_{y}^{2}b_{\prime}^{2}$. We have

$$D_{x}^{2}b'^{2} = 2b'e''' + c''^{2},$$
$$D_{:}D_{x}^{2}b'^{2} = 2b'e_{\prime}'' + 2b_{\prime}e''' + 2c''c_{\prime}',$$
$$D_{:}^{2}D_{x}^{2}b'^{2} = 2b'e_{\prime\prime}' + 2b_{\prime}e_{\prime}'' + 2c''c_{\prime\prime} + c_{\prime}'^{2},$$
$$D_{:}^{3}D_{x}^{2}b'^{2} = 2b'e_{\prime\prime\prime} + 2b_{\prime}e_{\prime\prime}' + 2c_{\prime}'c_{\prime\prime},$$
$$D_{:}^{4}D_{x}^{2}b'^{2} = 2b_{\prime}e_{\prime\prime\prime} + c_{\prime\prime}^{2} = D_{y}^{2}b_{\prime}^{2}.$$

The easiest mode of preparing for the development of a function is to write down the coefficients of $\phi'a$, $\frac{1}{2}\phi''a$, &c. in the coefficients of the simple powers of x, and then to perform as many $D_{y:x}$ derivations as there are in the exponent of the power, the test of correctness being the ultimate production of the D_{y} derivations in the coefficients of the simple powers of y. The following is as much of an instance as the page will allow of; it is for the terms of the third order.

	$\phi'a$	$\dfrac{\phi''a}{2}$	$\dfrac{\phi'''a}{2.3}$
x^3	$D_{x}^{2}b' = e'''$	$D_{x}b'^{2} = 2b'c''$	b'^{3}
x^2y	$e_{\prime}''$	$2b'c_{\prime}' + 2b_{\prime}c''$	$3b'^{2}b_{\prime}$
xy^2	$e_{\prime\prime}'$	$2b'c_{\prime\prime} + 2b_{\prime}c_{\prime}'$	$3b'b_{\prime}^{2}$
y^3	$e_{\prime\prime\prime}$	$2b_{\prime}c_{\prime\prime}$	$b_{\prime}^{3}$

and the complete exhibition of the terms of the third order is

$$\left(\phi'a.e''' + \frac{\phi''a}{2}\,2b'c'' + \frac{\phi'''a}{2.3}\,b'^{3}\right)x^{3}$$
$$+\left(\phi'a.e_{\prime}'' + \frac{\phi''a}{2}\,\{2b'c_{\prime}' + 2b_{\prime}c''\} + \frac{\phi'''a}{2.3}\,3b'^{2}b_{\prime}\right)x^{2}y$$
$$+\left(\phi'a.e_{\prime\prime}' + \frac{\phi''a}{2}\,\{2b'c_{\prime\prime} + 2b_{\prime}c_{\prime}'\} + \frac{\phi'''a}{2.3}\,3b'b_{\prime}^{2}\right)xy^{2}$$
$$+\left(\phi'a.e_{\prime\prime\prime} + \frac{\phi''a}{2}\,.\,2b_{\prime}c_{\prime\prime} \qquad + \frac{\phi'''a}{2.3}\,b_{\prime}^{3}\right)y^{3}.$$

To enable the reader to exercise himself further, I insert the terms of the fourth and fifth order, making ϕ_{n} stand for $\phi^{(n)}a \div 1.2.3..n$. The coefficients of

x^4, x^3y, x^2y^2, xy^3, y^4, are

$f^{\text{iv}}\phi_1 + (2b'e''' + c''^2)\,\phi_2 + 3b'^2c''\phi_3 + b'^4\phi_4$

$f_{\prime}'''\phi_1 + (2b'e_{\prime}'' + 2b_{\prime}e''' + 2c''c_{\prime}')\,\phi_2 + (3b'^2c_{\prime}' + 6b'b_{\prime}c'')\,\phi_3 + 4b'^3b_{\prime}\phi_4$

$f_{\prime\prime}''\phi_1 + (2b'e_{\prime\prime}' + 2b_{\prime}e_{\prime}'' + 2c''c_{\prime\prime} + c_{\prime}'^2)\,\phi_2 + (3b'^2c_{\prime\prime} + 6b'b_{\prime}c_{\prime}' + 3b_{\prime}^2c'')\,\phi_3 + 6b'^2b_{\prime}^2\phi_4$

$f_{\prime\prime\prime}'\phi_1 + (2b'e_{\prime\prime\prime} + 2b_{\prime}e_{\prime\prime}' + 2c_{\prime}'c_{\prime\prime})\,\phi_2 + (6b'b_{\prime}c_{\prime\prime} + 3b_{\prime}^2c_{\prime}')\,\phi_3 + 4b'b_{\prime}^3\phi_4$

$f_{\text{iv}}\phi_1 + (2b_{\prime}e_{\prime\prime\prime} + c_{\prime\prime}^2)\,\phi_2 + 3b_{\prime}^2c_{\prime\prime}\phi_3 + b_{\prime}^4\phi_4.$

The coefficients of x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5, are

$g^{\text{v}}\phi_1 + (2b'f^{\text{iv}} + 2c''e''')\,\phi_2 + (3b'^2e''' + 3b'c''^2)\,\phi_3 + 4b'^3c''\phi_4 + b'^5\phi_5$

$g_{\prime}^{\text{iv}}\phi_1 + (2b'f_{\prime}''' + 2b_{\prime}f^{\text{iv}} + 2c''e_{\prime}'' + 2c_{\prime}'e''')\,\phi_2 + (3b'^2e_{\prime}'' + 6b'b_{\prime}e''' + 6b'c''c_{\prime}' + 3b_{\prime}c''^2)\,\phi$

$\quad + (4b'^3c_{\prime}' + 12b'^2b_{\prime}c'')\,\phi_4 + 5b'^4b_{\prime}\phi_5$

$g_{\prime\prime}'''\phi_1 + (2b'f_{\prime\prime}'' + 2b_{\prime}f_{\prime}''' + 2c''e_{\prime\prime}' + 2c_{\prime}'e_{\prime}'' + 2c_{\prime\prime}e''')\,\phi_2$

$\quad + (3b'^2e_{\prime\prime}' + 6b'b_{\prime}e_{\prime}'' + 3b_{\prime}^2e''' + 6b'c''c_{\prime\prime} + 3b'c_{\prime}'^2 + 6b_{\prime}c''c_{\prime}')\,\phi_3$

$\quad + (4b'^3c_{\prime\prime} + 12b'^2b_{\prime}c_{\prime}' + 12b'b_{\prime}^2c'')\,\phi_4 + 10b'^3b_{\prime}^2\phi_5$

$g_{\prime\prime\prime}''\phi_1 + (2b'f'_{\prime\prime\prime} + 2b_{\prime}f_{\prime\prime}'' + 2c''e_{\prime\prime\prime} + 2c_{\prime}'e_{\prime\prime}' + 2c_{\prime\prime}e_{\prime}'')\,\phi_2$

$\quad + (3b'^2e_{\prime\prime\prime} + 6b'b_{\prime}e_{\prime\prime}' + 3b_{\prime}^2e_{\prime}'' + 6b'c_{\prime}'c_{\prime\prime} + 6b_{\prime}c''c_{\prime\prime} + 3b_{\prime}c_{\prime}'^2)\,\phi_3$

$\quad + (12b'^2b_{\prime}c_{\prime\prime} + 12b'b_{\prime}^2c_{\prime}' + 4b_{\prime}^3c'')\,\phi_4 + 10b'^2b_{\prime}^3\phi_5$

$g'_{\text{iv}}\phi_1 + (2b'f_{\text{iv}} + 2b_{\prime}f'_{\prime\prime\prime} + 2c_{\prime}'e_{\prime\prime\prime} + 2c_{\prime\prime}e'_{\prime\prime})\,\phi_2 + (6b'b_{\prime}e_{\prime\prime\prime} + 3b_{\prime}^2e'_{\prime\prime} + 3b'c_{\prime\prime}^2 + 6b_{\prime}c_{\prime}'c_{\prime\prime})\,\phi_3$

$\quad + (12b'b_{\prime}^2c_{\prime\prime} + 4b_{\prime}^3c_{\prime}')\,\phi_4 + 5b'b_{\prime}^4\phi_5$

$g_{\text{v}}\phi_1 + (2b_{\prime}f_{\text{iv}} + 2c_{\prime\prime}e_{\prime\prime\prime})\,\phi_2 + (3b_{\prime}^2e_{\prime\prime\prime} + 3b_{\prime}c_{\prime\prime}^2)\,\phi_3 + 4b_{\prime}^3c_{\prime\prime}\phi_4 + b_{\prime}^5\phi_5.$

§ 4. *On Functions of two series of one Variable.*

Having $V = \phi(a, A)$, it is proposed to develope

$$\phi(a + bx + cx^2 + \ldots,\quad A + Bx + Cx^2 + \ldots).$$

If we write this with two different variables, as

$$\phi(a + bx + \ldots,\quad A + By + \ldots),$$

and if we first write $a + bx + \ldots$ for a in $\phi(a, A)$ or V, the development is $V + D_x V \,.\, x + D_x^2 V \,.\, x^2 + \ldots$ If in each of the functions V, $D_x V$, &c. we write $A + By + \ldots$ for A, we find for the development required,

$$V + D_x V.x + D_y V.y + D_x^2 V.x^2 + D_x D_y V.xy + D_y^2 V.y^2 + \ldots$$

Change y into x, and the coefficient of x^n is

$$D_x^n V + D_x^{n-1} D_y V + \ldots + D_x D_y^{n-1} V + D_y^n V.$$

If the sum of the partial derivatives with respect to a, b, &c. and A, B, &c. be called the total derivative, and denoted by $D_{\prime\prime}$, we have

$$\phi(a, A) + D_{\prime\prime}\,\phi(a, A)\,x + D_{\prime\prime}^2\,\phi(a, A)\,.\,x^2 + \ldots$$

for the development. To expand this form, observe that in

$$D_x^m\,\phi(a, A) = \phi'\,D^{m-1}b + \frac{\phi''}{2}\,.\,D^{m-2}b^2 + \ldots + \frac{\phi^{(m)}}{2\,\ldots\,m}\,b'^m,$$

if we perform the operation D_y, it can affect nothing except the factors ϕ', ϕ'', &c. And if we adopt the notation $V_{m,n}$ to signify

$$\frac{d^{m+n}V}{da^m\,dA^n} \times \frac{1}{2.3\ldots m} \times \frac{1}{2.3\ldots n},$$

we have

$$\begin{aligned} D_x^{\,m} D_y^{\,n} V &= V_{1,1} D^{m-1} b\, D^{n-1} B + \ldots \\ &= \Sigma V_{p,q}\, D_x^{\,n-p} b^p \,.\, D_y^{\,n-q} B^q, \end{aligned}$$

for every pair of values of p and q, in which the former lies between 1 and m, the latter between 1 and n, both inclusive. Now $D_x^{\,k} b^p \,.\, D_y^{\,l} B^q$ is not distinguishable from $D_x^{\,k} D^{\,l} (b^p B^q)$. If we collect the expressions for $D_x^{\,n} V$, $D_x^{\,n-1} D_y V$, &c. up to $D_y^{\,n} V$, we find that $V_{p,q}$ occurs in every form in which $p+q$ does not exceed n, and that the coefficient of $V_{p,q}$ is

$$D_{,,}^{\,n-p-q}(b^p B^q), \quad \text{or} \quad b^p D_y^{\,n-p-q} B^q + D_x b^p D_y^{\,n-p-q-1} B^q + \ldots$$

So that the coefficient of x^n is

$$\begin{aligned} &V_{1,0}\, D_{,,}^{\,n-1} b + V_{0,1}\, D_{,,}^{\,n-1} B \\ &+ V_{2,0}\, D_{,,}^{\,n-2} b^2 + V_{1,1}\, D_{,,}^{\,n-2} . bB + V_{0,2}\, D^{n-2} B^2 \\ &+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ &+ V_{n,0}\, b^n + V_{n-1,1}\, b^{n-1} B + \ldots + V_{1,n-1}\, b\, B^{n-1} + V_{0,n}\, B^n, \end{aligned}$$

in which the $D_{,,}$ derivations require further development.

This paper has extended to such a length, that I will not make any further remark except the following. Arbogast's methods in general must be valued more than in proportion to the extent of development which they are used to obtain. For terms of the first and second orders they save no trouble, and very little for terms of the third order. From thence upwards they continue saving a larger and larger fraction of the time and labour which the common methods require. Independently of any value which the extension of the principle of differentiation in development may be found to have in high analysis, the insight which these methods give into the structure of algebraical expressions, and the power which they add to operation, render them deserving of much more attention than they have received.

University College, London, Feb. 16, 1846.

ON SYMBOLICAL GEOMETRY.

By Sir William Hamilton.

[*Continued.*]

On the Distributive Character of the Operation of Multiplication, as performed generally on Geometrical Fractions.

14. We are now prepared to extend the formulæ (76), (77), respecting the multiplication of sums of geometrical fractions; and to shew that similar results hold good, even when the condition of colinearity, assumed in those two formulæ, is no longer supposed to be satisfied. That is, the two equations

$$\left(\frac{h}{g}+\frac{f}{e}\right)\times\frac{k}{i}=\left(\frac{h}{g}\times\frac{k}{i}\right)+\left(\frac{f}{e}\times\frac{k}{i}\right)\ldots\ldots(104),$$

$$\frac{k}{i}\times\left(\frac{h}{g}+\frac{f}{e}\right)=\left(\frac{k}{i}\times\frac{h}{g}\right)+\left(\frac{k}{i}\times\frac{f}{e}\right)\ldots\ldots(105),$$

can both be shown to be true, whatever may be the lengths and directions of the six lines e, f, g, h, i, k; although, by the general non-commutativeness of geometrical fractions as factors, which was pointed out in the last article, the expressions contained in these two equations are not to be confounded with each other.

Making for this purpose

$$\left.\begin{array}{c}\frac{f}{e}=\beta_1+b_1,\quad \frac{h}{g}=\beta_2+b_2,\quad \frac{k}{i}=a+\alpha,\\ I\beta_1'\parallel I\alpha,\quad I\beta_1''\perp I\alpha,\quad I\beta_1''+I\beta_1'=I\beta_1,\\ I\beta_2'\parallel I\alpha,\quad I\beta_2''\perp I\alpha,\quad I\beta_2''+I\beta_2'=I\beta_2,\\ \beta_2'+\beta_1'=\beta',\ \beta_2''+\beta_1''=\beta'',\ \beta_2+\beta_1=\beta,\ b_2+b_1=b,\end{array}\right\}\ldots(106),$$

the conditions (83) will be satisfied; and if we still assign to γ and c the meanings (87), the equation (88) will hold good, and $\gamma + c$ will be an expression for the first member of (104). Making also, in imitation of (87),

$$\left.\begin{array}{l}c_1=\beta_1'a+b_1a,\quad \gamma_1=\beta_1''a+\beta_1\alpha+b_1\alpha,\\ c_2=\beta_2'a+b_2a,\quad \gamma_2=\beta_2''a+\beta_2\alpha+b_2\alpha,\end{array}\right\}\ldots(107),$$

the second member of the same equation (104) becomes, by the principles of the 11th article, $(\gamma_2+c_2)+(\gamma_1+c_1)$; and the equation resolves itself into the two following,

$$c=c_2+c_1,\quad \gamma=\gamma_2+\gamma_1\ldots\ldots\ldots(108);$$

which are easily seen to reduce themselves to these two,

$$(\beta_2'+\beta_1')\,a=\beta_2'a+\beta_1'a;\quad (\beta_2''+\beta_1'')\,a=\beta_2''a+\beta_1''a..(109);$$

the one being an equation between scalars, and the other between vectors. In like manner the equation (105) may be shown to depend on the two following equations, less general than itself, but of the same form,

$$\alpha(\beta_2' + \beta_1') = \alpha\beta_2' + \alpha\beta_1'\,; \quad \alpha(\beta_2'' + \beta_1'') = \alpha\beta_2'' + \alpha\beta_1'' \ldots (110).$$

And since, by (101), the three scalar products in the equations (110) are respectively equal, and the three vector products are respectively opposite (in their signs) to the corresponding products in the equations (109), it is sufficient to prove either of these two pairs of equations; for example, the pair (110). Now the first equation of this pair is true, because the scalars denoted by the three products $\alpha\beta_1'$, $\alpha\beta_2'$, $\alpha(\beta_2' + \beta_1')$, are proportional, both in their magnitudes and in their signs, to the indices of the three parallel vectors β_1', β_2', $\beta_2' + \beta_1'$; and the second equation of the same pair is true, because the indices of the vectors denoted by the three other products $\alpha\beta_1''$, $\alpha\beta_2''$, $\alpha(\beta_2'' + \beta_1'')$ are formed from the indices of the three coplanar vectors β_1'', β_2'', $\beta_2'' + \beta_1''$, by causing the three latter indices to revolve together, as one system, in their common plane, round the index Iα, their lengths being at the same time changed (if at all) in one common ratio, namely, in that of $\bar{\alpha}$ to 1. The formulæ (104) (105) are therefore proved to be true; and the same reasoning shows, that in any multiplication of two geometrical fractions, either of the factors may be *distributed* into *any number* of parts, and that the sum of the partial products will be equal to the total product: so that we may write, generally,

$$\left(\Sigma\frac{\mathrm{k}}{\mathrm{i}}\right) \times \left(\Sigma\frac{\mathrm{f}}{\mathrm{e}}\right) = \Sigma\left(\frac{\mathrm{k}}{\mathrm{i}} \times \frac{\mathrm{f}}{\mathrm{e}}\right) \ldots\ldots (111).$$

The *multiplication of geometrical fractions* is therefore *a distributive operation;* although it has been shown to be *not,* in general, a *commutative* one.

On the Associative Property of the Multiplication of Geometrical Fractions.

15. Proceeding now, with the help of the distributive property established in the last article, and of the principle that a product is multiplied by a scalar when any one of its factors is multiplied thereby, to prove that the multiplication of geometrical fractions is generally an *associative* operation, or that the formula

$$\frac{k}{i} \times \left(\frac{h}{g} \times \frac{f}{e}\right) = \left(\frac{k}{i} \times \frac{h}{g}\right) \times \frac{f}{e} \;\ldots\ldots\; (112),$$

holds good for *any three fractions* (with other formulæ of the same sort for more fractional factors than three), it will be sufficient to prove that the formula is true for *any three vectors;* or that we may write generally

$$\gamma \times \beta\alpha = \gamma\beta \times \alpha \;\ldots\ldots\ldots\ldots (113),$$

the vector γ being not here obliged to satisfy the equation (87); we may even content ourselves with proving that the equation (113) is true in the two following cases, namely first, when any two of the three vectors are parallel; and secondly, when all three are rectangular to each other. The first case may be expressed by the three following equations as its types—

$$\beta \times \beta\alpha = \beta\beta \times \alpha \;\ldots\ldots\ldots\; (114),$$

$$\beta \times \alpha\beta = \beta\alpha \times \beta \;\ldots\ldots\ldots\; (115),$$

$$\alpha \times \beta\beta = \alpha\beta \times \beta \;\ldots\ldots\ldots\; (116);$$

and the second case may be expressed by the equation

$$\alpha\beta \times \beta\alpha = (\alpha\beta \times \beta) \times \alpha, \quad \text{when } \beta \perp \alpha \;..\; (117);$$

because, under this last condition, $\alpha\beta$ is, by Art. 13, a vector, rectangular to both α and β. Under the same condition we may, by (99), change $\alpha\beta$ to $-\beta\alpha$; therefore the first member of the equation (117) may be equated to $-(\beta\alpha)^2$, and consequently, by (96), to $(-\beta^2) \times (-\alpha^2) = \beta^2 \times \alpha^2 = \beta^2\alpha \times \alpha = (\alpha \times \beta\beta) \times \alpha$, because β^2 or $\beta\beta$ is, by Art. 12, a scalar; thus we may make (117) depend on (116), which again depends on (114), and on the following equation,

$$\beta \times \beta\alpha = \alpha\beta \times \beta \;\ldots\ldots\ldots\; (118).$$

Equations (118) and (115) may both be proved by observing that, by Art. 13, whatever two vectors may be denoted by α and β, we have the expressions

$$\left.\begin{aligned} \beta\alpha &= \mathrm{S}\,.\,\beta\alpha + \mathrm{V}\,.\,\beta\alpha, \\ \alpha\beta &= \mathrm{S}\,.\,\beta\alpha - \mathrm{V}\,.\,\beta\alpha, \end{aligned}\right\} \;\ldots\ldots\; (119),$$

with the relations

$$\left.\begin{aligned} \beta \times \mathrm{S}\,.\,\beta\alpha - \mathrm{S}\,.\,\beta\alpha \times \beta &= 0, \\ \beta \times \mathrm{V}\,.\,\beta\alpha + \mathrm{V}\,.\,\beta\alpha \times \beta &= 0, \end{aligned}\right\} \;\ldots\; (120).$$

It remains then to prove the equation (114); and it is sufficient to prove this for the case where α and β are two rectangular vectors. But, in this case, $\beta\alpha$ is a vector formed from α by causing its index $\mathrm{I}\alpha$ to revolve right-handedly through a right angle round the index $\mathrm{I}\beta$, to which it is

perpendicular, changing at the same time in general the length of this revolving index from $\bar{a}$ to $\bar{\beta} \times \bar{a}$; and the repetition of this process, directed by the symbol $\beta \times \beta a$, conducts to a new vector, of which the index is in direction opposite to the original direction of Ia, and in length equal to $\bar{\beta}^2 \times \bar{a}$: this new vector may therefore be otherwise denoted by $-\bar{\beta}^2 \times a$, or by $\beta^2 \times a$, and the equation (114) is true. The equations (113) and (112) are therefore also true; and since the latter formula may easily be extended to any number of fractional factors, we are now entitled to conclude what it was at the beginning of the present article proposed to prove; namely, that the *multiplication of geometrical fractions is always an associative operation:* as the addition of fractions, and the addition of lines, have in former articles been shown to be. In other words, any number of successive fractional factors may be *associated* or grouped together by multiplication (without altering their order) into a single product, and this product substituted as a single factor in their stead; a result which constitutes a new agreement (the more valuable on account of the absence of identity in some other important respects), between the *rules of operation* of ordinary algebra, and those of the present Symbolical Geometry.

Other forms of the Associative Principle of Multiplication.

16. By the principles already established respecting the transformation of geometrical fractions, any three such fractions, $\frac{f}{e}$, $\frac{h}{g}$, $\frac{k}{i}$, may be so prepared that the numerator of the first shall be in the plane of the second, and that the numerator of the second shall coincide with the denominator of the third; we may, therefore, without diminishing the generality of the theorem expressed by the formula (112), suppose that the line i is equal to h, and that the fourth proportional to g, h, f, is a new line l; and with this preparation the associative principle of multiplication, established in the foregoing article, may be put under the following form, in which the mark of multiplication between two fractional factors is omitted for the sake of conciseness:

$$\text{if } \frac{h}{g} = \frac{l}{f}, \quad \text{then } \frac{k}{h}\,\frac{l}{e} = \frac{k}{g}\,\frac{f}{e} \;\ldots.\; (121);$$

that is to say, *the product of any two geometrical fractions will remain unaltered* in value, or will still continue to repre-

sent the same third fraction, *if the denominator of the multiplier and the numerator of the multiplicand be changed to any two new lines to which they are proportional,* or with which they form a *symbolic analogy,* including a relation between *directions* as well as a proportion of lengths, of the kind considered in Mr. Warren's work, (and earlier by Argand and Français,) and in the seventh article of this paper. Reciprocally, by the associative principle, the former of the two equations (121) is in general a consequence of the latter; that is, if the product of two geometrical fractions be equal to the product of two other fractions of the same sort, and if the multipliers have a common numerator, and the multiplicands a common denominator, then the numerators of the two multiplicands and the denominators of the two multipliers are the antecedents and consequents of a symbolical proportion or analogy, of the kind considered in the seventh article: for we may write

$$\frac{h}{g} = \frac{h}{k}\left(\frac{k}{g}\,\frac{f}{e}\right)\frac{e}{f}, \quad \frac{h}{k}\left(\frac{k}{h}\,\frac{l}{e}\right)\frac{e}{f} = \frac{l}{f};$$

so that the first equation (121) may be obtained from the second, by suitably grouping or associating the factors.

Again, the same associative principle shows that

$$\text{if } \frac{c}{c'} = \frac{b'}{b}\,\frac{a'}{a}, \quad \text{then } \frac{c}{b'} = \frac{c'}{a}\,\frac{a'}{b} \ldots\ldots (122);$$

for the first equation (122) may be replaced by the system of the three following equations,

$$\frac{a'}{a} = \frac{b''}{a''}, \quad \frac{b'}{b} = \frac{c''}{b''}, \quad \frac{c'}{c} = \frac{a''}{c''} \ldots\ldots (123);$$

of which the two last give, for the first member of the second equation (122), the expression

$$\frac{c}{b'} = \frac{c'}{a''}\,\frac{b''}{b},$$

which is equal to the second member of the same second equation (122), by the first of the three equations (123), and by the theorem (121): whenever, therefore, we meet an equation between one geometrical fraction and the product of two others, we are at liberty to *interchange the denominator of the product and the numerator of the multiplier,* provided that we at the same time *interchange the denominators of the two factors;* no change being made in the numerators of the product and the multiplicand. Conversely, this assertion respecting the liberty to make these interchanges, and the

formula (122), to which the assertion corresponds, are modes of expressing the associative principle of multiplication; for by introducing the equations (123) we find that the theorem (122) conducts to the following relation, or *identity between the two ternary products of three fractions*, associated in two different ways, but with one common order of arrangement,

$$\frac{c'}{a''}\left(\frac{a''}{a}\,\frac{a'}{b}\right)=\left(\frac{c'}{a''}\,\frac{a''}{a}\right)\frac{a'}{b}\,\ldots\ldots\ldots\,(124);$$

in which last form, as in (112), the three factors multiplied together may represent any three geometrical fractions. We may also present the same principle under the form of the following theorem—

$$\text{if }\frac{c'}{c}\,\frac{b'}{b}\,\frac{a'}{a}=1,\quad\text{then }\frac{c'}{a}\,\frac{a'}{b}\,\frac{b'}{c}=1\,\ldots\,(125);$$

and may derive from it, with the help of (123), the following value of a certain product of six fractional factors,

$$\frac{a''}{c''}\,\frac{c}{a}\,\frac{b''}{a''}\,\frac{a}{b}\,\frac{c''}{b''}\,\frac{b}{c}=1\,\ldots\ldots\ldots\,(126):$$

which must hold good whenever the three lines a, b, c are respectively coplanar with the three pairs a″b″, b″c″, c″a″. Finally, it may be stated here, as a theorem essentially equivalent to the associative principle of multiplication, although not expressly involving any product of two or more fractions, that *in the system of the six equations* of which those marked (123) are three, and of which the others are the three following analogous equations,

$$\frac{a}{c'}=\frac{a'''}{c'''},\quad\frac{b}{a'}=\frac{b'''}{a'''},\quad\frac{c}{b'}=\frac{c'''}{b'''}\,\ldots\ldots(127);$$

any five equations of the system include the sixth.

Geometrical Interpretation of the Associative Principle: Symbolic Equations between Arcs upon a Sphere: Theorem of the two Spherical Hexagons.

17. If we attended only to the *lengths* of the various lines compared, the associative principle of multiplication, under all the foregoing forms, would be nothing more than an easy and known consequence of a few elementary theorems respecting compositions of ratios of magnitudes. On the other hand it is permitted, in the present symbolical geometry, to assume at pleasure the *situations* of straight lines denoted by small roman letters, provided that the lengths and directions are preserved. The general theorem or property of

multiplication, which has been expressed in various ways in the two foregoing articles, may therefore be regarded as being essentially a *relation, or system of relations, between the directions of certain lines in space.*

In this view of the subject no essential loss of generality (or at least none which cannot easily be supplied by known and elementary principles) will be sustained by supposing all the straight lines abc, $a'b'c'$, $a''b''c''$, $a'''b'''c'''$, e f g h i k l, of the two last articles to be *radii of one sphere,* setting out from one *common origin* or centre O, and terminating in points upon one *common spheric surface,* which may be denoted respectively by the symbols ABC, $A'B'C'$, $A''B''C''$, $A'''B'''C'''$, EFGHIKL. In order more conveniently to study and express relations between points so situated, we may agree to say that two *arcs upon one sphere,* such as those from G to H and from F to L, are *symbolically equal,* when they are *equally long and similarly directed portions of the circumference of one great circle;* and may denote this *symbolical equality between arcs,* so called for the sake of suggesting that (like the symbolical equality between straight lines considered in the second article) it involves a relation of *identity of directions,* as well as a relation of equality of lengths, by writing any one of the three formulæ,

$$\left.\begin{array}{l} \frown LF = \frown HG, \\ \frown FL = \frown GH, \\ \frown LH = \frown FG, \end{array}\right\} \ldots\ldots\ldots\ldots (128);$$

of which the second may be called the *inverse,* and the third the *alternate* of the first. Any one of these three formulæ (128) will thus express the *same relation between the directions of four coplanar radii,* namely, the four lines f g h l, as that expressed by the first equation (121), or by its inverse, or its alternate equation; that is, by any one of the three following equations between geometrical fractions,

$$\frac{l}{f} = \frac{h}{g}, \quad \frac{f}{l} = \frac{g}{h}, \quad \frac{l}{h} = \frac{f}{g} \ldots\ldots\ldots (129).$$

The formulæ (128) express also the same relation between the same four directions, as that which would be expressed in a notation of a former article, by any one of the three following *symbolic analogies* between the same four lines,

$$l : f :: h : g, \quad f : l :: g : h, \quad l : h :: f : g \ldots (130);$$

although it must not be forgotten that any one of the six latter formulæ, (129) and (130), expresses at the same time a

proportion between the lengths of four straight lines, not generally equal to each other, which is not expressed by any one of the three former symbolical equations (128), between pairs of arcs upon a sphere. In this notation (128), the last form of the associative principle of multiplication which was assigned in the foregoing article, so far as it relates to directions only, may be expressed by saying that *any one of the six following symbolical equations between arcs is a consequence of the other five*,

$$\left.\begin{array}{l} \frown A'A = \frown B''A'', \\ \frown B'B = \frown C''B'', \\ \frown C'C = \frown A''C'', \end{array}\right\} \ldots\ldots\ldots (131);$$

$$\left.\begin{array}{l} \frown BA' = \frown B'''A''', \\ \frown CB' = \frown C'''B''', \\ \frown AC' = \frown A'''C''', \end{array}\right\} \ldots\ldots\ldots (132).$$

Regarding *any six points* upon a spheric surface, in *any one order* of succession, as the *six corners of a spherical hexagon* (which may have re-entrant angles, and of which two or more sides may cross each other without being prolonged), we may speak of the arcs joining *successive corners* as the *sides;* those joining *alternate* corners, as the *diagonals;* and those joining *opposite* corners, as the *diameters* of this hexagon: the first side, first diagonal, and first diameter, respectively, being those three arcs which are drawn from the first corner to the second, third, and fourth corners of the figure. With this phraseology, the form just now obtained for the result of the two foregoing articles may be expressed as a relation between two spherical hexagons, $AA'BB'CC'$, $A''A'''B'''B''C''C'''$, and may be enunciated in words as follows: *If five successive sides of one spherical hexagon be respectively and symbolically equal to five successive diagonals of another spherical hexagon, the sixth side of the first hexagon will be symbolically equal to the sixth diagonal of the second hexagon.* This theorem of spherical geometry, which may be called, for the sake of reference, the *theorem of the two hexagons*, is therefore a consequence, and may be regarded as an interpretation of the associative principle of multiplication: and conversely, in all applications to spherical geometry, and generally in all investigations respecting relations between the directions of straight lines in space, the associative principle of multiplication may be replaced by the theorem of the two spherical hexagons.

[*To be continued.*]

ON THE ROTATION OF A SOLID BODY ROUND A FIXED POINT.

By Arthur Cayley.

(*Continued from* p. 173.)

On the Variation of the Constants, when the body is acted upon by forces.

The dynamical equations of a problem being expressed in the form

$$\frac{d}{dt}\cdot\frac{dT}{d\lambda'}-\frac{dT}{d\lambda}=\frac{dV}{d\lambda},$$

$$\frac{d}{dt}\cdot\frac{dT}{d\mu'}-\frac{dT}{d\mu}=\frac{dV}{d\mu}.$$

$$\frac{d}{dt}\cdot\frac{dT}{d\nu'}-\frac{dT}{d\nu}=\frac{dV}{d\nu}.$$

Suppose the equations obtained from these by neglecting the function V, are integrated; each of the six integrals may be expressed in the form

$$a=f(\lambda, \mu, \nu, \lambda', \mu', \nu', t),$$

where a denotes any one of the arbitrary constants. Assume

$$\frac{dT}{d\lambda'}=u, \quad \frac{dT}{d\mu'}=v, \quad \frac{dT}{d\nu}=w.$$

Then λ', μ', ν' may be expressed in terms of $\lambda, \mu, \nu, u, v, w$, and the integrals may be reduced to the form

$$a=F(\lambda, \mu, \nu, u, v, w, t).$$

These equations may be considered as the integrals of the proposed system, taking into account the terms involving V, provided $a, b, \ldots$ &c. be supposed to become variable. We have, in this case, by Lagrange's theory of the variation of the arbitrary constants, the formulæ

$$\frac{da}{dt}=(a,b)\frac{dV}{db}+(a,c)\frac{dV}{dc}+(a,d)\frac{dV}{dd}+(a,e)\frac{dV}{de}+(a,f)\frac{dV}{df};$$

where

$$(a,b)=\left(\frac{da}{du}\frac{db}{d\lambda}-\frac{da}{d\lambda}\frac{db}{du}\right)+\left(\frac{da}{dv}\frac{db}{d\mu}-\frac{da}{d\mu}\frac{db}{dv}\right)+\left(\frac{da}{dw}\frac{db}{d\nu}-\frac{da}{d\nu}\frac{db}{dw}\right),$$

and in which V is supposed to be expressed as a function of a, b, c, d, e, f, t.

Thus the solution of the problem requires the calculation of thirty coefficients (a, b), or rather of fifteen only, since evidently $(a, b) = -(b, a)$. It is known that these coefficients are functions of a, b, c, d, e, f, without t; so that, in calculating them, any assumed arbitrary value, e.g. $t = 0$, may be given to the time.

In practice, it often happens that one of the arbitrary constants, e.g. (a), may be expressed in the form

$$a = F(\lambda, \mu, \nu, u, v, w, t, b, c, d, e, f),$$

where b, c, d, e, f are given functions of $\lambda, \mu, \nu, u, v, w, t$. In this case, it is easily seen that we may write

$$(a, b) = \{(a, b)\} + (c, b)\frac{da}{dc} + (d, b)\frac{da}{dd} + (e, b)\frac{da}{de} + (f, b)\frac{da}{df},$$

where, in the calculation of $\{(a, b)\}$, the differentiations upon a are performed, without taking into account the variability of $b, c \ldots.$

In the particular problem in question, the following are the values of the new variables u, v, w, (*Math. Journal*, memoir already quoted),

$$u = \frac{2}{\kappa}(Ap - \nu Bq + \mu Cr) \ldots\ldots (29),$$

$$v = \frac{2}{\kappa}(\nu Ap + Bq - \lambda Cr),$$

$$w = \frac{2}{\kappa}(-\mu Ap + \lambda Bq + Cr);$$

equations which may also be expressed in the form

$$\begin{aligned} 2Ap &= (1 + \lambda^2)\,u + (\lambda\mu + \nu)\,v + (\nu\lambda - \mu)\,w \ldots .(30), \\ 2Bq &= (\lambda\mu - \nu)\,u + (1 + \mu^2)\,v + (\mu\nu + \lambda)\,w, \\ 2Cr &= (\nu\lambda + \mu)\,u + (\mu\nu - \lambda)\,v + (1 + \nu^2)\,w, \end{aligned}$$

or putting for shortness

$$\lambda u + \mu v + \nu w = \varpi \ldots\ldots\ldots\ldots .(31),$$

these become

$$\begin{aligned} 2Ap &= \lambda\varpi + u + \nu v - \mu w \ldots\ldots\ldots (32), \\ 2Bq &= \mu\varpi - \nu u + v + \lambda w, \\ 2Cr &= \nu\varpi + \mu u - \lambda v + w. \end{aligned}$$

Whence also

$$2\Omega = \kappa\varpi \ldots\ldots\ldots\ldots\ldots\ldots .(33).$$

Substituting the values of Ap, Bq, Cr, given by (30) in the equations (6), we deduce

$$2\mathrm{a} = \lambda\varpi + u - \nu v + \mu w \ldots\ldots\ldots\ldots (34),$$
$$2\mathrm{b} = \mu\varpi + \nu u + v - \lambda w,$$
$$2\mathrm{c} = \nu\varpi - \mu u + \lambda v + w,$$

whence also $$2(\mathrm{a}\lambda + \mathrm{b}\mu + \mathrm{c}\nu) = \kappa\varpi \ldots\ldots\ldots\ldots (35),$$

which in fact follows from (33) and (17). And likewise the inverse system,

$$u = \frac{2}{\kappa}(\mathrm{a} + \nu\mathrm{b} - \mu\mathrm{c}) \ldots\ldots\ldots\ldots (36),$$
$$v = \frac{2}{\kappa}(-\nu\mathrm{a} + \mathrm{b} + \lambda\mathrm{c}),$$
$$w = \frac{2}{\kappa}(\mu\mathrm{a} - \lambda\mathrm{b} + \mathrm{c}).$$

It is easy to deduce

$$k^2 = \frac{1}{4}\kappa\,[u^2 + v^2 + w^2 + \varpi^2] \ldots\ldots\ldots\ldots (37),$$
$$\upsilon = \frac{1}{4}\,[(u^2 + v^2 + w^2) + (1 + \kappa)\,\varpi^2] \ldots\ldots (38).$$

Again, from the equations (10 *bis*),

$$\kappa(\mathrm{b}Cr - \mathrm{c}Bq) = -2\lambda(\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2) + 2\mathrm{a}(\lambda\mathrm{a} + \mu\mathrm{b} + \nu\mathrm{c}) + 2(\mathrm{b}\nu - \mathrm{c}\mu)\,\Omega$$
$$= -2\lambda k^2 + 2(\mathrm{a} + \mathrm{b}\nu - \mathrm{c}\mu)\,\Omega$$
$$= -2\lambda k^2 + \kappa u\Omega\,;$$

[by equations (36),] *i.e.*

$$\Omega u - \frac{2}{\kappa}k^2\lambda = \mathrm{b}Cr - \mathrm{c}Bq \ldots\ldots\ldots\ldots (39),$$
$$\Omega v - \frac{2}{\kappa}k^2\mu = \mathrm{c}Ap - \mathrm{b}Cr,$$
$$\Omega w - \frac{2}{\kappa}k^2\nu = \mathrm{a}Bq - \mathrm{c}Ap\,;$$

to which many others might probably be joined.

The constants of the problem are a, b, c, h, ϵ, δ. Of these a, b, c are given as functions of λ, μ, ν, u, v, w, by the equations (34); in which ϖ is to be considered as standing for $\lambda u + \mu v + \nu w$. [These determine k^2, which is however given immediately by (37).] As for h, we have

$$h = \frac{1}{A}(Ap)^2 + \frac{1}{B}(Bq)^2 + \frac{1}{C}(Cr)^2 \ldots\ldots (40),$$

where Ap, Bq, Cr are given as functions of λ, μ, ν, u, v, w by (32), in which also ϖ stands for $\lambda u + \mu v + \nu w$. Again,

$$\epsilon = t - \frac{1}{2}\int \frac{dv}{\nabla},$$

$$\delta = 2 \tan^{-1} \frac{\kappa\varpi}{2k} - \frac{1}{4} k \int \frac{(h + \Phi)\, dv}{v\nabla}.$$

In each of which ∇, Φ are functions of v, and of a, b, c, h, partly as entering explicitly into these functions, partly as contained implicitly in p, q, r, which enter into ∇, Φ, and are functions of v, h, k given by (18). After the integration v is to be considered a function of λ, μ, ν, u, v, w given by (38). Both of the integrals may be supposed taken from a certain value v_0 of v, which may be considered as an absolutely invariable arbitrary constant, since without it we have the right number, six, of arbitrary constants. First to find (a, b), (b, c), and (c, a). From (34) we have

$$\begin{aligned}(\mathrm{a}, \mathrm{b}) = \tfrac{1}{4}\{&(1 + \lambda^2)(\mu u - w) - (\lambda u + \varpi)(\lambda\mu + \nu)\\ &+ (\lambda\mu - \nu)(\mu v + \varpi) - (\lambda v + w)(1 + \mu^2)\\ &+ (\nu\lambda + \mu)(u + \mu w) - (\lambda w - v)(\mu\nu - \lambda)\}\\ = \tfrac{1}{2}&(\mu u - \lambda v - w - \nu\varpi) = -\tfrac{1}{2}2\mathrm{c} = -\mathrm{c};\end{aligned}$$

whence the system

$$(\mathrm{b}, \mathrm{c}) = -\mathrm{a}, \quad (\mathrm{c}, \mathrm{a}) = -\mathrm{b}, \quad (\mathrm{a}, \mathrm{b}) = -\mathrm{c} \;\ldots\; (40).$$

Also we may add

$$(k, \mathrm{a}) = \frac{\mathrm{a}}{k}(\mathrm{a}, \mathrm{a}) + \frac{\mathrm{b}}{k}(\mathrm{b}, \mathrm{a}) + \frac{\mathrm{c}}{k}(\mathrm{c}, \mathrm{a}) = 0,$$

or $$(k, \mathrm{a}) = 0, \quad (k, \mathrm{b}) = 0, \quad (k, \mathrm{c}) = 0 \;\ldots\ldots\; (41),$$

which will be useful in calculating some of the following coefficients.

Proceeding to calculate (a, h), (b, h), (c, h). It is seen immediately that

$$(\mathrm{a}, h) = 2\{p(\mathrm{a}, Ap) + q(\mathrm{a}, Bq) + r(\mathrm{a}, Cr)\},$$

where Ap, Bq, Cr, are given by the equations (32), so that

$$\begin{aligned}(\mathrm{a}, Ap) = \tfrac{1}{4}\{&(1 + \lambda^2)(\lambda u + \varpi) - (1 + \lambda^2)(\lambda u + \varpi)\\ &+ (\lambda\mu - \nu)(\lambda v - w) - (\lambda\mu + \nu)(\lambda v + w)\\ &+ (\nu\lambda + \mu)(v + \lambda w) - (\nu\lambda - \mu)(-v + \lambda w)\}\end{aligned}$$

i.e. $(\mathrm{a}, Ap) = 0.$ .... (42).

Similarly

$$\begin{aligned}(\mathrm{a}, Bq) = \tfrac{1}{4}\{&(1+\lambda^2)(\mu u + w) - (\lambda u + \varpi)(\lambda\mu - \nu)\\ &+(\lambda\mu - \nu)(\mu v + \varpi) - (\lambda v + w)(1+\mu^2)\\ &+(\nu\lambda + \mu)(\mu w - u) - (-v + \lambda w)(\mu\nu + \lambda)\}\end{aligned}$$

i.e. $(\mathrm{a}, Bq) = 0$, and similarly ...... (43),
$(\mathrm{a}, Cr) = 0$;

whence $(\mathrm{a}, h) = 0$, and $\therefore (\mathrm{b}, h) = 0$, $(\mathrm{c}, h) = 0$ ...... (44);
also $(k, h) = 0$, ... (45).

Next we have to determine (a, ϵ), (b, ϵ), (c, ϵ). Here ϵ being a function of $u, v, w, \lambda, \mu, \nu$, a, b, c, h, we must write

$$(\mathrm{a}, \epsilon) = \{(\mathrm{a}, \epsilon)\} + (\mathrm{a}, \mathrm{b})\frac{d\epsilon}{d\mathrm{b}} + (\mathrm{a}, \mathrm{c})\frac{d\epsilon}{d\mathrm{c}} + (\mathrm{a}, h)\frac{d\epsilon}{dh},$$

i.e. $(\mathrm{a}, \epsilon) = \{(\mathrm{a}, \epsilon)\} + \mathrm{b}\dfrac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\dfrac{d\epsilon}{d\mathrm{b}}$.

But $$\epsilon = t - 2\int\frac{dv}{\nabla};$$

whence $$\{(\mathrm{a}, \epsilon)\} = -\frac{2}{\nabla}(\mathrm{a}, v),$$

and v is given immediately as a function of $\lambda, \mu, \nu, u, v, w$, by the equation (38). Hence

$$\begin{aligned}(\mathrm{a}, v) &= \tfrac{1}{4}\left[(1+\lambda^2)\{(1+\kappa)u\varpi + \lambda\varpi^2\} - (\lambda u + \varpi)\{u + \lambda(1+\kappa)\varpi\}\right.\\ &\quad + (\lambda\mu - \nu)\{(1+\kappa)v\varpi + \mu\varpi^2\} - (\lambda v + w)\{v + \mu(1+\kappa)\varpi\}\\ &\quad \left.+ (\nu\lambda + \mu)\{(1+\kappa)w\varpi + \nu\varpi^2\} - (-v + \lambda w)\{w + \nu(1+\kappa)\varpi\}\right]\\ &= \tfrac{1}{4}\{(1+\kappa)\varpi u - \lambda(1+\kappa)\varpi^2 + \lambda\kappa - \lambda(u^2+v^2+w^2) - u\varpi\}\\ &= \tfrac{1}{4}\{\kappa u\varpi - \lambda\varpi^2 - \lambda(u^2+v^2+w^2)\}\\ &= \tfrac{1}{4}\kappa u\varpi - \frac{k^2\lambda}{\kappa} = \tfrac{1}{2}\left(\Omega u - \frac{2k^2\lambda}{\kappa}\right) \text{ [by (37) and (33)]},\\ &= \tfrac{1}{2}(\mathrm{b}Cr - \mathrm{c}Bq). \qquad \text{...... (46);}\end{aligned}$$

whence $$\{(\mathrm{a}, \epsilon)\} = -\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq),$$

$$(\mathrm{a}, \epsilon) = -\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) + \mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}}.$$

The terms $\mathrm{b}\dfrac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\dfrac{d\epsilon}{d\mathrm{b}}$ are evidently of the form $F(v) - F(v_0)$. If therefore we suppose $v = v_0$, we have

$$(\mathrm{a}, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0) \text{ .... (47)},$$

if p_0, q_0, r_0, ∇_0 refer to the value v_0 of v, i.e. if

$$Ap_0^2 + Bq_0^2 + Cr_0^2 = h \dots\dots\dots\dots (48),$$
$$A^2p_0^2 + B^2q_0^2 + C^2r_0^2 = k^2,$$
$$Ap_0\mathrm{a} + Bq_0\mathrm{b} + Cr_0\mathrm{c} = 2v_0 - k^2.$$

[This implies evidently

$$\mathrm{b}\frac{d\epsilon}{dc} - \mathrm{c}\frac{d\epsilon}{db} = \frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) - \frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0),$$

an equation which it is interesting to verify. In fact, from the value of ϵ

$$\mathrm{b}\frac{d\epsilon}{d\mathrm{c}} - \mathrm{c}\frac{d\epsilon}{d\mathrm{b}} = -2\int dv\left(\mathrm{b}\frac{d}{d\mathrm{c}} - \mathrm{c}\frac{d}{d\mathrm{b}}\right)\frac{1}{\nabla} = 2\int dv\,\frac{1}{\nabla^2}\left(\mathrm{b}\frac{d\nabla}{d\mathrm{c}} - \mathrm{c}\frac{d\nabla}{d\mathrm{b}}\right);$$

or we have to shew that

$$\frac{d}{dv}\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) = \frac{2}{\nabla^2}\left(\mathrm{b}\frac{d\nabla}{d\mathrm{c}} - \mathrm{c}\frac{d\nabla}{d\mathrm{b}}\right) = \frac{2}{\nabla^2}\delta\nabla;$$

if for shortness,

$$\delta = \mathrm{b}\frac{d}{d\mathrm{c}} - \mathrm{c}\frac{d}{d\mathrm{b}}.$$

Now ∇ containing a, b, c explicitly, and also as involved in p, q, r, we have

$$\delta\nabla = \mathrm{b}pq(A-B) - \mathrm{c}rp(C-A) + \frac{d\nabla}{dp}\delta p + \frac{d\nabla}{dq}\delta q + \frac{d\nabla}{dr}\delta r$$
$$= \mathrm{b}pq(A-B) - \mathrm{c}rp(C-A) + \delta'\nabla$$

suppose. The equation to be verified becomes

$$\nabla\left(\mathrm{b}C\frac{dr}{dv} - \mathrm{c}B\frac{dq}{dv}\right) - (\mathrm{b}Cr - \mathrm{c}Bq)\frac{d\nabla}{dv}$$
$$= 2\{\mathrm{b}pq(A-B) - \mathrm{c}rp(C-A) + \delta'\nabla\}.$$

Now, observing that $\delta k = 0$, we have

$$Ap\delta p + Bq\delta q + Cr\delta r = 0,$$
$$A^2p\delta p + B^2q\delta q + C^2r\delta r = 0,$$
$$A\mathrm{a}\delta p + B\mathrm{b}\delta q + C\mathrm{c}\delta r = -(\mathrm{b}Cr - \mathrm{c}Bq).$$

Also,

$$Ap\frac{dp}{dv} + Bq\frac{dq}{dv} + Cr\frac{dr}{dv} = 0,$$
$$A^2p\frac{dp}{dv} + B^2q\frac{dq}{dv} + C^2r\frac{dr}{dv} = 0,$$
$$A\mathrm{a}\frac{dp}{dv} + B\mathrm{b}\frac{dq}{dv} + C\mathrm{c}\frac{dr}{dv} = 2.$$

Whence evidently

$$\frac{dp}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta p, \quad \frac{dq}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta q, \quad \frac{dr}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta r,$$

or

$$\frac{d\nabla}{dv} = \frac{-2}{\mathrm{b}Cr - \mathrm{c}Bq}\,\delta'\nabla\,;$$

or the equation to be verified is simply

$$\nabla\left(\mathrm{b}C\frac{dr}{dv} - \mathrm{c}B\frac{dq}{dv}\right) = 2\left\{\mathrm{b}pq\,(A - B) - \mathrm{c}\,rp\,(C - A)\right\};$$

which follows immediately from the three equations just given for the determination of $\frac{dp}{dv}, \frac{dq}{dv}, \frac{dr}{dv}$].

From which values also

$$(k, \epsilon) = 0 \ldots\ldots\ldots\ldots\ldots (49).$$

Next, to calculate (h, ϵ),

$$(h, \epsilon) = \{(h, \epsilon)\} + (h, \mathrm{a})\frac{d\epsilon}{d\mathrm{a}} + (h, \mathrm{b})\frac{d\epsilon}{d\mathrm{b}} + (h, \mathrm{c})\frac{d\epsilon}{d\mathrm{c}}.$$

But the three last terms being evidently such as to vanish for $v = v_0$, we may neglect them, and consider (h, ϵ) as the value which $\{(h, \epsilon)\}$ assumes for this value of v.

Now $\{(h, \epsilon)\} = 2p\,\{(Ap, \epsilon)\} + 2q\,\{(Bq, \epsilon)\} + 2r\,\{(Cr, \epsilon)\}$,

where $\quad \{(Ap, \epsilon)\} = -\frac{2}{\nabla}(Ap, v),$

and

$$\begin{aligned}(Ap, v) = {}&\tfrac{1}{4}\big[(1+\lambda^2)\{(1+\kappa)u\varpi + \lambda\varpi^2\} - (\lambda u + \varpi)\{u + \lambda(1+\kappa)\varpi\}\\ &+ (\lambda\mu + \nu)\{(1+\kappa)v\varpi + \mu\varpi^2\} - (\lambda v - w)\{v + \mu(1+\kappa)\varpi\}\\ &+ (\nu\lambda - \mu)\{(1+\kappa)w\varpi + \nu\varpi^2\} - (v + \lambda w)\{w + \nu(1+\kappa)\varpi\}\big]\\ = {}&\tfrac{1}{4}\{(1+\kappa)\varpi u + \lambda\kappa\varpi^2 - \lambda(1+\kappa)\varpi^2 - \lambda(u^2+v^2+w^2) - \varpi u\} = (\mathrm{a}, v)\\ = {}&\tfrac{1}{2}(\mathrm{b}Cr - \mathrm{c}Bq), \qquad \ldots\ldots (50),\end{aligned}$$

whence $\quad \{(Ap, \epsilon)\} = -\frac{1}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) \ldots\ldots\ldots (51),$

and therefore $\quad \{(Bq, \epsilon)\} = -\frac{1}{\nabla}(\mathrm{c}Ap - \mathrm{a}Cr),$

$$\{(Cr, \epsilon)\} = -\frac{1}{\nabla}(\mathrm{a}Bq - \mathrm{b}Ap),$$

whence $\quad \{(h, \epsilon) = -2,$

and therefore $\quad (h, \epsilon) = -2 \ldots\ldots\ldots\ldots\ldots\ldots (52).$

Next, to find (a, δ), (b, δ), (c, δ), (h, δ),

$$\delta = 2\tan^{-1}\frac{\kappa\varpi}{2k} - k\int\frac{(h+\Phi)\,dv}{v\nabla}$$

$$= \delta' + \delta'' \quad \text{suppose},$$

$$(\mathrm{a}, \delta) = (\mathrm{a}, \delta') + (\mathrm{a}, \delta''),$$

$$(\mathrm{a}, \delta') = \frac{k}{\kappa v}(\mathrm{a}, \kappa\varpi) + (\mathrm{a}, k)\frac{d\delta'}{dk}$$

[observing $\kappa^2\varpi^2 + 4k^2 = 4(\Omega^2 + k^2) = 4\kappa v$]

$$= \frac{k}{\kappa v}(\mathrm{a}, \kappa\varpi),$$

where

$$\begin{aligned}(\mathrm{a}, \kappa\varpi) = \tfrac{1}{2}\{&(1+\lambda^2)(\kappa u + 2\lambda\varpi) - (\lambda u + \varpi)\kappa\lambda \\ &+ (\lambda\mu - \nu)(\kappa v + 2\mu\varpi) - (\lambda v + w)\kappa\mu \\ &+ (\nu\lambda + \mu)(\kappa w + 2\nu\varpi) - (-v + \lambda w)\kappa\nu\} \\ = \tfrac{1}{2}\kappa(u + \lambda\varpi) &= Ap - \nu Bq + \mu Cr + \lambda\Omega = \tfrac{1}{2}(\mathrm{a} + Ap)\kappa \ldots (53),\end{aligned}$$

by equations (29), (33), and (10);

or $$(\mathrm{a}, \delta') = \frac{k}{2v}(\mathrm{a} + Ap).$$

Also $$(\mathrm{a}, \delta'') = -k\frac{h+\Phi}{v\nabla}(\mathrm{a}, v) + (\mathrm{a}, \mathrm{b})\frac{d\delta''}{d\mathrm{b}} + \&\mathrm{c}.$$

$$= -\tfrac{1}{2}k\frac{h+\Phi}{v\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) + Fv - Fv_0,$$

whence

$$(\mathrm{a}, \delta) = \frac{k}{2v}\left\{\mathrm{a} + Ap - \frac{h+\Phi}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq)\right\} + Fv - Fv_0,$$

or putting $v = v_0$,

$$(\mathrm{a}, \delta) = \frac{k}{2v_0}\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\} \ldots\ldots (54),$$

and therefore

$$(\mathrm{b}, \delta) = \frac{k}{2v_0}\left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\},$$

$$(\mathrm{c}, \delta) = \frac{k}{2v_0}\left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}.$$

Again, $(h, \delta) = 2p(Ap, \delta) + 2q(Bq, \delta) + 2r(Cr, \delta)$,

$$(Ap, \delta) = (Ap, \delta') + (Ap, \delta''),$$

$$(Ap, \delta') = \frac{k}{\kappa v}(Ap, \kappa\varpi) + (Ap, k)\frac{d\delta'}{dk}$$

$$= \frac{k}{\kappa v}(Ap, \kappa\varpi),$$

$$(Ap, \kappa\varpi) = \tfrac{1}{2}\{(1+\lambda^2)(\kappa u + 2\lambda\varpi) - (\lambda u + \varpi)\kappa\lambda$$
$$+ (\lambda\mu + \nu)(\kappa v + 2\mu\varpi) - (\lambda v - w)\kappa\mu$$
$$+ (\nu\lambda + \mu)(\kappa w + 2\nu\varpi) - (v + \lambda w)\kappa\nu\}$$
$$= \tfrac{1}{2}\kappa(u + \lambda\varpi) = \tfrac{1}{2}\kappa(\mathrm{a} + Ap) \quad \ldots\ldots\ldots\ldots (55);$$

$$\therefore (Ap, \delta') = \frac{k}{2u}(\mathrm{a} + Ap) \quad \ldots\ldots\ldots\ldots (56),$$

$$(Ap, \delta'') = -k\frac{h+\Phi}{\upsilon\nabla}(Ap, \upsilon) + \&\text{c.}$$
$$= -\tfrac{1}{2}k\frac{h+\Phi}{\upsilon\nabla}(\mathrm{b}Cr - \mathrm{c}Bq) + F\upsilon - F\upsilon_0,$$
$$(Ap, \delta) = \frac{k}{2\upsilon}\left\{\mathrm{a} - Ap - \frac{h+\Phi}{\nabla}(\mathrm{b}Cr - \mathrm{c}Bq)\right\} + F\upsilon - F\upsilon_0,$$

and similarly for (Bq, δ), (Cr, δ). Substituting, and neglecting the terms which vanish for $\upsilon = \upsilon_0$,

$$(h, \delta) = \frac{k}{\upsilon}\left(\Phi + h - \frac{\Phi + h}{\nabla}\nabla\right),$$

$$\text{i.e.} \quad (h, \delta) = 0 \quad \ldots\ldots\ldots\ldots (57).$$

Lastly, to find (ϵ, δ),

$$(\epsilon, \delta) = \{(\epsilon, \delta)\} + (\mathrm{a}, \delta)\frac{d\epsilon}{d\mathrm{a}} + (\mathrm{b}, \delta)\frac{d\epsilon}{d\mathrm{b}} + (\mathrm{c}, \delta)\frac{d\epsilon}{d\mathrm{c}},$$

where, in $\{(\epsilon, \delta)\}$, the differentiations upon ϵ are supposed not to affect the constants a, b, c. Neglecting the terms which vanish for $\upsilon = \upsilon_0$,

$$(\epsilon, \delta) = \{(\epsilon, \delta)\},$$
$$\{(\epsilon, \delta)\} = \{(\epsilon, \delta')\} + \{(\epsilon, \delta'')\},$$
$$\{(\epsilon, \delta')\} = [\{(\epsilon, \delta')\}] + (\epsilon, k)\frac{d\delta'}{dk} = [\{(\epsilon, \delta')\}];$$

where, in $[\{(\epsilon, \delta')\}]$, the differentiations upon ϵ and δ do not affect the constants.

$$\{(\epsilon, \delta'')\} = [\{(\epsilon, \delta'')\}] + (\epsilon, \mathrm{a})\frac{d\delta''}{d\mathrm{a}} + \&\text{c.}$$
$$\text{i.e.} \quad \{(\epsilon, \delta'')\} = [\{(\epsilon, \delta'')\}]:$$

neglecting the terms which vanish for $\upsilon = \upsilon_0$,

$$\therefore (\epsilon, \delta) = [\{(\epsilon, \delta')\}] + [\{(\epsilon, \delta'')\}]$$
$$= [\{(\epsilon, \delta')\}];$$

since $$[\{(\epsilon, \delta'')\}] = (\upsilon, \upsilon)\frac{d\epsilon}{d\upsilon}\,\frac{d\delta''}{d\upsilon} = 0.$$

Hence $$(\epsilon, \delta) = -\tfrac{1}{2}\frac{k}{\kappa\nabla\upsilon}(\upsilon, \kappa\varpi)\,\ldots\ldots\ldots\ldots(58),$$

$$\begin{aligned}(\upsilon, \kappa\varpi) &= \tfrac{1}{2}\left\{\begin{aligned}&\{u+(1+\kappa)\lambda\varpi\}(2\lambda\varpi+\kappa u)-\{\lambda\varpi^2+(1+\kappa)\varpi u\}\kappa\lambda\\ &+\{v+(1+\kappa)\mu\varpi\}(2\mu\varpi+\kappa v)-\{\mu\varpi^2+(1+\kappa)\varpi v\}\kappa\mu\\ &+\{w+(1+\kappa)\nu\varpi\}(2\nu\varpi+\kappa w)-\{\nu\varpi^2+(1+\kappa)\varpi w\}\kappa\nu\end{aligned}\right\}\\ &= \tfrac{1}{2}\{2\varpi^2+\kappa(u^2+v^2+w^2)+2(1+\kappa)(\kappa-1)\varpi^2\\ &\qquad +\kappa(1+\kappa)\varpi^2-\kappa(\kappa-1)\varpi^2-\kappa(\kappa+1)\varpi^2\}\\ &= \tfrac{1}{2}\kappa\{(\kappa+1)\varpi^2+(u^2+v^2+w^2)\} = \tfrac{1}{2}4\kappa\upsilon = 2\kappa\upsilon\,\ldots(59),\end{aligned}$$

therefore

$$(\epsilon, \delta) = -\frac{k}{\nabla_0}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(60).$$

Hence, recapitulating,

$$\left.\begin{aligned}&(\mathrm{b}, \mathrm{c}) = -\mathrm{a}, \quad (\mathrm{c}, \mathrm{a}) = -\mathrm{b}, \quad (\mathrm{a}, \mathrm{b}) = -\mathrm{c},\\ &(\mathrm{a}, h) = 0, \quad (\mathrm{b}, h) = 0, \quad (\mathrm{c}, h) = 0,\\ &(\mathrm{a}, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0),\\ &(\mathrm{b}, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0),\\ &(\mathrm{c}, \epsilon) = -\frac{1}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0),\\ &(h, \epsilon) = -2,\\ &(\mathrm{a}, \delta) = \frac{k}{2\upsilon_0}\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\},\\ &(\mathrm{b}, \delta) = \frac{k}{2\upsilon_0}\left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\},\\ &(\mathrm{c}, \delta) = \frac{k}{2\upsilon_0}\left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\},\\ &(h, \delta) = 0,\\ &(\epsilon, \delta) = -\frac{k}{\nabla_0},\end{aligned}\right\}\ldots(61),$$

and therefore

$$\left.\begin{aligned}
\frac{d\mathrm{a}}{dt} &= -\mathrm{c}\frac{dV}{d\mathrm{b}} + \mathrm{b}\frac{dV}{d\mathrm{c}} - \frac{1}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\frac{dV}{d\epsilon} \\
&\quad + \frac{k}{2v_0}\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\}\frac{dV}{d\delta}, \\
\frac{d\mathrm{b}}{dt} &= -\mathrm{a}\frac{dV}{d\mathrm{c}} + \mathrm{c}\frac{dV}{d\mathrm{a}} - \frac{1}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\frac{dV}{d\epsilon} \\
&\quad + \frac{k}{2v_0}\left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\}\frac{dV}{d\delta}, \\
\frac{d\mathrm{c}}{dt} &= -\mathrm{b}\frac{dV}{d\mathrm{a}} + \mathrm{a}\frac{dV}{d\mathrm{b}} - \frac{1}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\frac{dV}{d\epsilon} \\
&\quad + \frac{k}{2v_0}\left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}\frac{dV}{d\delta}, \\
\frac{dh}{dt} &= -2\frac{dV}{d\epsilon}, \\
\frac{d\epsilon}{dt} &= \frac{1}{\nabla_0}\left\{(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\frac{dV}{d\mathrm{a}} + (\mathrm{c}Ap_0 - \mathrm{a}Bq_0)\frac{dV}{d\mathrm{b}}\right. \\
&\quad \left. + (\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\frac{dV}{d\mathrm{c}}\right\} + 2\frac{dV}{dh} - \frac{k}{\nabla_0}\frac{dV}{d\delta}, \\
\frac{d\delta}{dt} &= -\frac{k}{2v_0}\left[\left\{\mathrm{a} + Ap_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{b}Cr_0 - \mathrm{c}Bq_0)\right\}\frac{dV}{d\mathrm{a}}\right. \\
&\quad + \left\{\mathrm{b} + Bq_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{c}Ap_0 - \mathrm{a}Cr_0)\right\}\frac{dV}{d\mathrm{b}} \\
&\quad \left. + \left\{\mathrm{c} + Cr_0 - \frac{h+\Phi_0}{\nabla_0}(\mathrm{a}Bq_0 - \mathrm{b}Ap_0)\right\}\frac{dV}{d\mathrm{c}}\right] + \frac{k}{\nabla_0}\frac{dV}{d\epsilon},
\end{aligned}\right\} \quad \ldots\ldots (62),$$

to which we may join $\dfrac{dk}{dt} = \dfrac{dV}{d\delta}$ .... (63).

ON THE DIAMETRAL PLANES OF A SURFACE OF THE SECOND ORDER.

By Arthur Cayley.

Let $U = Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gxz + 2Hxy = 0$, be the equation of a surface of the second order referred to its centre, and let $ax + a'y + a''z = 0$ be the equation of one of its diametral planes; then, as usual,

$$(A-u)\,a + Ha' + Ga'' = 0,$$
$$Ha + (B-u)\,a' + Fa'' = 0,$$
$$Ga + Fa' + (C-u)\,a'' = 0,$$

which are equivalent to two independent equations, and consequently capable of determining the ratios $a : a' : a''$, provided that u satisfy the cubic equation that is obtained by eliminating a, a', a'' from the three equations.

We have from the second and third, from the third and first, and from the first and second equations respectively,

$$a : a' : a'' = \mathfrak{A}_{,} : \mathfrak{H}_{,} : \mathfrak{G}_{,} = \mathfrak{H}_{,} : \mathfrak{B}_{,} : \mathfrak{F}_{,} = \mathfrak{G}_{,} : \mathfrak{F}_{,} : \mathfrak{C}_{,}\,;$$

where, if

$$\mathfrak{A} = BC - F^2,$$
$$\mathfrak{B} = CA - G^2,$$
$$\mathfrak{C} = AB - H^2,$$
$$\mathfrak{F} = GH - AF,$$
$$\mathfrak{G} = HF - BG,$$
$$\mathfrak{H} = FG - CH.$$

$\mathfrak{A}_{,}$, $\mathfrak{B}_{,}$, $\mathfrak{C}_{,}$, $\mathfrak{F}_{,}$, $\mathfrak{G}_{,}$, $\mathfrak{H}_{,}$ are what these become when A, B, C are changed into $A-u$, $B-u$, $C-u$, so that

$$\mathfrak{A}_{,} = \mathfrak{A} - (B+C)\,u + u^2,$$
$$\mathfrak{B}_{,} = \mathfrak{B} - (C+A)\,u + u^2,$$
$$\mathfrak{C}_{,} = \mathfrak{C} - (A+B)\,u + u^2,$$
$$\mathfrak{F}_{,} = \mathfrak{F} + Fu,$$
$$\mathfrak{G}_{,} = \mathfrak{G} + Gu,$$
$$\mathfrak{H}_{,} = \mathfrak{H} + Hu.$$

Hence the equation $ax + a'y + a''z = 0$ may be written in the three forms

$$\mathfrak{A}_{,}x + \mathfrak{H}_{,}y + \mathfrak{G}_{,}z = 0,$$
$$\mathfrak{H}_{,}x + \mathfrak{B}_{,}y + \mathfrak{F}_{,}z = 0,$$
$$\mathfrak{G}_{,}x + \mathfrak{F}_{,}y + \mathfrak{C}_{,}z = 0\,;$$

or, what comes to the same thing, as follows,

$$\mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z + u\,(Ax + Hy + Gz) + vx = 0,$$
$$\mathfrak{H}x + \mathfrak{B}y + \mathfrak{F}z + u\,(Hx + By + Fz) + vy = 0,$$
$$\mathfrak{G}x + \mathfrak{F}y + \mathfrak{C}z + u\,(Gx + Fy + Cz) + vz = 0,$$

in which for shortness v has been written instead of

$$u^2 - (A+B+C)\,u.$$

The elimination of u, v from these equations gives a result $\Theta = 0$, where Θ is a homogeneous function of the third order

in x, y, z; and this equation, it is evident, must belong to the three diametral planes jointly, *i.e.* Θ must be the product of three linear factors, each of which equated to zero would correspond to a diametral plane. Thus the system of diametral planes is given by

$$\Theta = \begin{vmatrix} \mathfrak{A}x + \mathfrak{H}y + \mathfrak{G}z, & Ax + Hy + Gz, & x \\ \mathfrak{H}x + \mathfrak{B}y + \mathfrak{F}z, & Hx + By + Fz, & y \\ \mathfrak{G}x + \mathfrak{F}y + \mathfrak{C}z, & Gx + Fy + Cz, & z \end{vmatrix} = 0,$$

or developing the determinant, as follows,

$$\begin{aligned}\Theta = {} & (G\mathfrak{H} - H\mathfrak{G})\, x^3 + (H\mathfrak{F} - F\mathfrak{H})\, y^3 + (F\mathfrak{G} - G\mathfrak{F})\, z^3 \\ & + \{G(\mathfrak{C} - \mathfrak{B}) - \mathfrak{G}(C - B) - (H\mathfrak{F} - F\mathfrak{H})\}\, yz^2 \\ & + \{H(\mathfrak{A} - \mathfrak{C}) - \mathfrak{H}(A - C) - (F\mathfrak{G} - G\mathfrak{F})\}\, zx^2 \\ & + \{F(\mathfrak{B} - \mathfrak{A}) - \mathfrak{F}(B - A) - (G\mathfrak{H} - H\mathfrak{G})\}\, xy^2 \\ & + \{- H(\mathfrak{C} - \mathfrak{B}) + \mathfrak{H}(C - B) + (F\mathfrak{G} - G\mathfrak{F})\}\, y^2 z \\ & + \{- F(\mathfrak{A} - \mathfrak{C}) + \mathfrak{F}(A - C) + (G\mathfrak{H} - H\mathfrak{G})\}\, z^2 x \\ & + \{- G(\mathfrak{B} - \mathfrak{A}) + \mathfrak{C}(B - A) + (H\mathfrak{F} - F\mathfrak{H})\}\, x^2 y \\ & + (C\mathfrak{B} - B\mathfrak{A} + \mathfrak{A}\mathfrak{C} - C\mathfrak{A} + B\mathfrak{A} - A\mathfrak{B})\, xyz;\end{aligned}$$

or reducing

$$\begin{aligned}\Theta = {} & \{F(G^2 - H^2) - GH(C - B)\}\, x^3 \\ & + \{G(H^2 - F^2) - HF(A - C)\}\, y^3 \\ & + \{H(F^2 - G^2) - FG(B - A)\}\, z^3 \\ & + \{G(A - B)(B - C) + FH(A + B - 2C) \\ & \qquad + G(F^2 + G^2 - 2H^2)\}\, yz^2 \\ & + \{H(B - C)(C - A) + GF(B + C - 2A) \\ & \qquad + H(G^2 + H^2 - 2F^2)\}\, zx^2 \\ & + \{F(C - A)(A - B) + GH(C + A - 2B) \\ & \qquad + F(H^2 + F^2 - 2G^2)\}\, xy^2 \\ & + \{H(B - C)(C - A) + FG(C + A - 2B) \\ & \qquad + H(H^2 + F^2 - 2G^2)\}\, y^2 z \\ & + \{F(C - A)(A - B) + GH(A + B - 2C) \\ & \qquad + F(F^2 + G^2 - 2H^2)\}\, z^2 x \\ & + \{G(A - B(B - C) + HF(B + C - 2A) \\ & \qquad + G(G^2 + H^2 - 2F^2)\}\, x^2 y \\ & - \{(A - B)(B - C)(C - A) \\ & \qquad + (B - C)F^2 + (C - A)G^2 + (A - B)H^2\}\, xyz.\end{aligned}$$

In the case of *curves* of the second order, the result is much simpler; we have

$$\Theta = \begin{vmatrix} Ax + Hy, & x \\ Hx + By, & y \end{vmatrix} = 0,$$

i.e. $$\Theta = H(y^2 - x^2) + (A - B)\,xy = 0,$$

for the equation of the two diameters.

The above formulæ may be applied to the question of finding the diametral planes of the cone circumscribed about a given surface of the second order, (or of the lines bisecting the angles made by two tangents of a curve of the second order). Considering the latter question first: if

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

be the equation of the curve, and α, β the coordinates of the point of intersection of the two tangents, the equation of the pair of tangents is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right) - \left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} - 1\right)^2 = 0;$$

or making the point of intersection the origin,

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - 1\right) - \left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2}\right)^2 = 0,$$

i.e. $$(\beta x - \alpha y)^2 - (b^2x^2 + a^2y^2) = 0;$$

whence $A = \beta^2 - b^2$, $B = \alpha^2 - a^2$, $H = -\alpha\beta$, and the equation to the lines bisecting the angles formed by the tangents is

$$\alpha\beta\,(x^2 - y^2) - \{\alpha^2 - \beta^2 - (a^2 - b^2)\}\,xy = 0,$$

which is the same for all confocal ellipses; whence the known theorem,

"If there be two confocal ellipses, and tangents be drawn to the second from any point P of the first, the tangent and normal of the first conic at the point P, bisect the angles formed by the two tangents in question."

In the case of surfaces, the equation of the circumscribing cone referred to its vertex as origin, is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right) - \left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2}\right)^2 = 0;$$

whence
$$\begin{aligned}
A &= \beta^2c^2 + \gamma^2b^2 - b^2c^2,\\
B &= \gamma^2a^2 + \alpha^2c^2 - a^2c^2,\\
C &= \alpha^2b^2 + \beta^2a^2 - b^2a^2,\\
F &= -\,a^2\beta\gamma,\\
G &= -\,b^2\gamma\alpha,\\
H &= -\,c^2\alpha\beta.
\end{aligned}$$

And thence omitting the factor $b^2c^2\alpha^2 + c^2a^2\beta^2 + a^2b^2\gamma^2 - a^2b^2c^2$,

$$\begin{aligned}
\mathfrak{A} &= \alpha^2 - a^2,\\
\mathfrak{B} &= \beta^2 - b^2,\\
\mathfrak{C} &= \gamma^2 - c^2,\\
\mathfrak{F} &= \beta\gamma,\\
\mathfrak{G} &= \gamma\alpha,\\
\mathfrak{H} &= \alpha\beta\,;
\end{aligned}$$

and the equation of the system of diametral planes becomes

$$\begin{aligned}
\Theta = 0 = x^3.\alpha^2\beta\gamma\,(c^2-b^2) &+ y^3.\beta^2\gamma\alpha\,(a^2-c^2) + z^3.\gamma^2\alpha\beta\,(b^2-a^2)\\
&+ \gamma\alpha\,\{\alpha^2(c^2-b^2) + \beta^2(b^2+c^2-2a^2) - \gamma^2(b^2-a^2)\\
&\qquad + (b^2-a^2)(c^2-b^2)\}\,yz^2\\
&+ \alpha\beta\,\{-\alpha^2(c^2-b^2) + \beta^2(a^2-c^2) + \gamma^2(c^2+a^2-2b^2)\\
&\qquad + (c^2-b^2)(a^2-c^2)\}\,zx^2\\
&+ \gamma\alpha\,\{\alpha^2(a^2+b^2-2c^2) - \beta^2(a^2-c^2) + \gamma^2(b^2-a^2)\\
&\qquad + (a^2-c^2)(b^2-a^2)\}\,xy^2\\
&- \alpha\beta\,\{\alpha^2(c^2-b^2) - \beta^2(a^2-c^2) - \gamma^2(b^2+c^2-2a^2)\\
&\qquad - (a^2-c^2)(c^2-b^2)\}\,y^2z\\
&- \beta\gamma\,\{-\alpha^2(c^2+a^2-2b^2) + \beta^2(a^2-c^2) - \gamma^2(b^2-a^2)\\
&\qquad - (b^2-a^2)(a^2-c^2)\}\,z^2x\\
&- \gamma\alpha\,\{-\alpha^2(c^2-b^2) - \beta^2(a^2+b^2-2c^2)\\
&\qquad + \gamma^2(b^2-a^2) - (c^2-b^2)(b^2-a^2)\}\,x^2y\\
&+ \{(a^2-b^2)(b^2-c^2)(c^2-a^2)\\
&\quad + (\alpha^4+\beta^2\gamma^2)(c^2-b^2) + (\beta^4+\gamma^2\alpha^2)(a^2-c^2) + (\gamma^4+\alpha^2\beta^2)(b^2-a^2)\\
&\quad + \alpha^2(b^2-c^2)(2a^2-b^2-c^2) + \beta^2(c^2-a^2)(2b^2-c^2-a^2)\\
&\qquad + \gamma^2(a^2-b^2)(2c^2-a^2-b^2)\}\,xyz.
\end{aligned}$$

And since this is a function of a^2-b^2, b^2-c^2, and c^2-a^2, the equation is the same for all confocal ellipsoids; whence the known theorem, "The axes of the circumscribing cone having its vertex in a given point P, are tangents to the curves of intersection of the three surfaces, confocal with the given surface, which pass through the point P."

SUR UNE PROPRIÉTÉ DE LA COUCHE ÉLECTRIQUE EN ÉQUILIBRE À LA SURFACE D'UN CORPS CONDUCTEUR.

Par *M.* J. Liouville.

La méthode la plus générale que l'on connaisse pour former des couches électriques, en équilibre à la surface de corps conducteurs, consiste à considérer une masse M; et le potentiel,

$$V = \iiint \frac{f(x', y', z')\, dx'\, dy'\, dz'}{\Delta},$$

de cette masse, par rapport à un point quelconque (x, y, z), dont la distance au point (x', y', z'), ou à l'élément

$$f(x', y', z')\, dx'\, dy'\, dz',$$

est désignée par Δ. Prenons ensuite une surface de niveau ou d'équilibre relativement à l'attraction de la masse M, et qui entoure cette masse, c'est à dire prenons une surface fermée (A), contenant la masse M dans son intérieur, et pour tous les points de laquelle V conserve une valeur constante. En fin soit $\frac{dV}{ds}\, ds$ la variation infiniment petite que V éprouve lorsqu' on passe d'un point de cette surface à un point extérieur infiniment voisin situé sur la normale à une distance ds. C'est la dérivée $\frac{dV}{ds}$, multipliée si l'on veut par une constante, qui réglera la loi des densités de l'électricité en équilibre sur un corps conducteur terminé par la surface (A). Plusieurs géomètres sont parvenus, chacun de leur côté, à ce beau théorême; mais c'est George Green qui l'a, je crois, donné le premier dans un excellent mémoire publié en 1828, sous ce titre: *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism.* Je me propose de montrer que la couche électrique en équilibre ainsi obtenue a précisément le même centre de gravité que la masse M.

Plaçons l'origine des coordonnées x, y, z, au centre de gravité de la masse M; et désignons par x_1 une quelconque des coordonnées du centre de gravité de la couche électrique, laquelle sera fournie par la formule

$$x_1 \iint \frac{dV}{ds}\, d\omega = \iint x\, \frac{dV}{ds}\, d\omega,$$

où les intégrations s'appliquent à la surface (A) dont l'élément est représenté par $d\omega$. Il s'agit de prouver que $x_1 = 0$.

D' après l'expression de V, on a

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi f(x, y, z), \quad \text{ou} = 0,$$

suivant que le point (x, y, z) appartient ou non à la masse M.

Pour plus de simplicité, écrivons toujours

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -4\pi f(x, y, z),$$

en regardant la fonction $f(x, y, z)$ comme nulle hors de la masse M; et combinons cette équation avec cette autre de forme analogue

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} = 0,$$

où nous supposons que U est une fonction de x, y, z, qui reste finie et continue ainsi que ses dérivées dans tout l'espace intérieur à (A). Nous aurons

$$V\frac{d^2U}{dx^2} - U\frac{d^2V}{dx^2} + V\frac{d^2U}{dy^2} - U\frac{d^2V}{dy^2} + V\frac{d^2U}{dz^2} - U\frac{d^2V}{dz^2}$$
$$= 4\pi U f(x, y, z).$$

Multiplions par $dx\,dy\,dz$, et intégrons dans tout l'espace intérieur à (A). En conservant à ds et à $d\omega$ la même signification que ci-dessus, on trouve, après des transformations bien connues:

$$\int\int V\frac{dU}{ds}\,d\omega - \int\int U\frac{dV}{ds}\,d\omega = 4\pi\int\int\int U f(x, y, z)\,dx\,dy\,dz.$$

Mais l'équation en U est satisfaite par $U = x$; nous avons donc:

$$\int\int V\frac{dx}{ds}\,d\omega - \int\int x\frac{dV}{ds}\,d\omega = 4\pi\int\int\int x f(x, y, z)\,dx\,dy\,dz.$$

L' intégrale triple du second membre, divisée par M, donne l'abscisse du centre de gravité de la masse M. Ce centre étant à l' origine des coordonnées, l' intégrale dont nous parlons est nulle. Je vais prouver que l' intégrale $\int\int V\frac{dx}{ds}\,d\omega$ l'est aussi. D' abord on peut faire sortir V du signe $\int$, puisque, sur la surface (A), V est constant. Observons ensuite que $\frac{dx}{ds}$ a pour valeur le cosinus de l' angle α que la normale ds fait avec l' axe des x. Notre intégrale

deviendra donc: $V\iint \cos a \, d\omega$. Or l'intégrale $\iint \cos a \, d\omega$ est nulle, d'après un théorème connu, comme composée d'éléments deux à deux égaux et de signes contraires. Ainsi $\iint V \frac{dx}{ds} d\omega = 0$. Il reste donc finalement

$$\iint x \frac{dV}{ds} d\omega = 0,$$

et l'on en conclut $x_1 = 0$, ce qu'il fallait démontrer.

TOUL, 4 *Juillet*, 1846.

NOTE ON THE PRECEDING PAPER.

By WILLIAM THOMSON.

[*Extracted from a Letter to* M. *Liouville.*]

"... THE demonstration which you have given has led me to this other theorem, that the mass M, and the shell surrounding it, have the same principal axes, through any point.

To demonstrate this, let $U = yz$ in the formula which you have given. Then, since, if we denote by K the constant value of V at the shell, we have

$$\iint V \frac{dU}{ds} d\omega = K \iint \frac{dU}{ds} = 0,^*$$

we find

$$\iint yz \frac{-dV}{ds} d\omega = 4\pi \iiint yz.f(x, y, z)\, dx\, dy\, dz \; \ldots\ldots \; (1),$$

which proves the proposition enunciated.

If we take $U = x^2$, we find

$$V\frac{d^2U}{dx^2} - U\frac{d^2V}{dx^2} + V\frac{d^2U}{dy^2} - U\frac{d^2V}{dy^2} + V\frac{d^2U}{dz^2} - U\frac{d^2V}{dz^2}$$
$$= 2V + 4\pi x^2 f(x, y, z):$$

from which, observing that

$$\iint V\left(\frac{dU}{dx} dy\,dz + \frac{dU}{dy} dz\,dx + \frac{dU}{dz} dx\,dy\right)$$
$$= K\iiint \left(\frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2}\right) dx\,dy\,dz$$
$$= 2K \iiint dx\,dy\,dz;$$

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* See First Series, Vol. III. p. 203, (8).

we deduce

$$\frac{1}{4\pi}\iint x^2 \frac{-dV}{ds}\, d\omega = \frac{1}{2\pi}\iiint (V-K)\, dx\, dy\, dz$$
$$+ \iiint x^2 f(x, y, z)\, dx\, dy\, dz.$$

Let A, B, C be the moments of inertia of the mass M round the axes of coordinates, and A_1, B_1, C_1 those of the shell, round the same axes, it being supposed that the quantity of matter of the shell is the same as that of M;* the preceding equation, and the two others which correspond relatively to the axes of y and z, are with this notation,

$$A_1 = Q + A,\; B_1 = Q + B,\; C_1 = Q + C \ldots\ldots (2),\dagger$$

where $$Q, = \frac{1}{2\pi}\iiint (V-K)\, dx\, dy\, dz,$$

is a quantity which is independent of the position of the origin.

From equations (2), we have

$$B - C = B_1 - C_1,\; C - A = C_1 - A_1,\; A - B = A_1 - B_1 \ldots (3).$$

A demonstration of your theorem and of the theorems expressed by the equations (1) and (3) may be arrived at by comparing the expressions for the equal potentials ‡ produced by the mass M, and the shell at very distant points."

St. Peter's College, July 15, 1846.

ACTION OF A FORCE WHOSE DIRECTION ROTATES IN A PLANE.

By Andrew Bell.

This paper treats of the motion of a physical point acted on by a constant force whose direction passes through the point, and has a uniform angular motion in one plane.

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* In this case the "density" of the distribution at any point of the shell will be equal to $\frac{1}{4\pi}\cdot\frac{-dV}{ds}$. See Vol. III., p. 75.

† If the origin be taken at the centre of gravity, and the axes of coordinates principal axes of M, (and therefore of the shell, according to the proposition enunciated above,) these equations shew that the "central ellipsoid" (see note to p. 202) for the shell is confocal with that for the body M.

‡ A shell constructed round the mass M, in the manner described by M. Liouville, with a quantity of matter equal to M, exerts the same force upon points without the shell, as was proved first by Green, (see also Vol. III., p. 75); and, since the potential of each vanishes at an infinite distance, it follows that the two bodies produce equal potentials at every point without the shell.

[If a rocket be made to revolve round a horizontal axis, and then, being ignited, be allowed to fall freely, its horizontal motion will be of the kind considered in this paper.]

Let Ox, Oy be rectangular axes in the same plane with the material point and with the direction of the force; x, y the co-ordinates of the point at any instant of time t, and ϕ the constant accelerating force making an angle θ with the axis x, the line of its direction revolving from right to left, or so as to increase θ, and let the force when $\theta < \frac{1}{2}\pi$, be directed from the origin, so as to increase the co-ordinates.

The resolved parts of ϕ parallel to the axes are respectively

$$P = \phi \cos \theta, \qquad Y = \phi \sin \theta.$$

If T be the time of revolution of the force, and t that of describing θ, then

$$2\pi : \theta = T : t; \quad \text{hence} \quad \theta = \frac{2\pi t}{T}:$$

or, if $\frac{2\pi}{T} = r$, then $\theta = rt$.

Hence the equations of motion now become

$$\frac{d^2x}{dt^2} = X, \qquad \frac{d^2y}{dt^2} = Y,$$

or

$$\frac{d^2x}{dt^2} = \phi \cos rt, \quad \frac{d^2y}{dt^2} = \phi \sin rt.$$

The first integrals of these equations are

$$\frac{dx}{dt} = \frac{\phi}{r} \sin rt + a, \qquad \frac{dy}{dt} = -\frac{\phi}{r} \cos rt + b.$$

Since these equations give the values of the velocities of the point in the directions of the axes, the constants will be determined by assigning values to these velocities at any instant. Let the components of its velocity in the directions of x and y be v' and v'' when $t = 0$; then will $a = v'$ and $b = v'' + \frac{\phi}{r}$, and consequently

$$\frac{dx}{dt} = \frac{\phi}{r} \sin rt + v', \quad \frac{dy}{dt} = \frac{\phi}{r}(1 - \cos rt) + v''.$$

Integrating again,

$$x = -\frac{\phi}{r^2} \cos rt + v't + a, \qquad y = \frac{\phi}{r^2}(rt - \sin rt) + v''t + b.$$

If the point is at the origin when $t = 0$, then

$$x = \frac{\phi}{r^2}(1 - \cos rt) + v't, \quad y = \frac{\phi}{r^2}(rt - \sin rt) + v''t.$$

Let m, n be such numbers that $v' = \frac{m}{r}\phi$, $v'' = \frac{n}{r}\phi$, then

$$x = \frac{\phi}{r^2}(mrt + 1 - \cos rt), \quad y = \frac{\phi}{r^2}\{(1 + n)\,rt - \sin rt\}.$$

This system of equations is that of a species of oblique cycloid, or rather of a series of such cycloids, the line of whose bases passes through the origin, and is expressed by the equation

$$x = \frac{m\phi}{r}\,y = cy, \quad \text{if } c = \frac{m\phi}{r}.$$

Since $1 - \cos rt$ is never negative, the value of x is never less than that obtained from the equation $x = cy$, so that no part of the series of equal curves lies between the line of the bases and the axes of y. The points in which the curves meet the line of the bases, will be found by assuming $rt = 2e\pi$, where e is any integer; for then

$$x = \frac{m\phi}{r}\,t = ct,$$

which indicates a point of this line. The successive points of meeting are found by giving e the consecutive integral values 1, 2, 3, ... The corresponding values of y are

$$y = 2e\,(1 + n)\,\pi\,\frac{\phi}{r^2}.$$

The axis is the greatest value of the absciss of the curve reckoned from the base line, and is

$$x' = \frac{2\phi}{r^2},$$

and the length of the oblique base projected on the ordinate is

$$= 2\,(1 + n)\,\pi\,\frac{\phi}{r^2}.$$

The type of the curve is the common, the curtate, or prolate cycloid, according as n is zero, negative, or positive.

If when $t = 0$, $v' = 0$, or the point have merely an initial motion in the direction of the axis y, this motion is represented by the system

$$x = \frac{\phi}{r^2}(1 - \cos rt), \quad y = \frac{\phi}{r^2}\{(1 + n)\,rt - \sin rt\}$$

These equations are the general equation of the cycloid, and belong to a series of equal cycloidal curves whose bases lie in the axis of y, and whose axes are parallel to the axis of x.

The length of the axis is $\frac{2\phi}{r^2}$, and of the base $2\pi(1+n)\frac{\phi}{r^2}$.

It can easily be proved, by determining when $\frac{dy}{dx}$ is $= 0$, or $= \infty$, that according as n is zero, positive, or negative, the curves are common, prolate, or curtate cycloids.

Since the last equation becomes that of the common cycloid when $n = 0$, it appears that the general equation to the cycloid is deducible from that of the common cycloid by adding to the value of y the term $\frac{n}{r}\phi t$, which is proportional to the time or to θ; consequently this term indicates the impression, on a point moving in a common cycloid, of a uniform velocity, in the direction of the base, and $= \frac{n}{r}\phi t$, in addition to the component due to its oscillation. Hence a common cycloidal pendulum may be made to oscillate in a prolate or curtate cycloid by impressing on it and its cycloidal cheeks an initial uniform velocity in the direction of its base, according as the direction of this velocity is towards the side to which the pendulum is to move, or towards the opposite side; the cheeks being constrained to retain their uniform velocity.

The result of this investigation establishes the fact, that a constant and uniformly rotating force is capable of producing a progressive motion, and it also affords another remarkable physical property of the cycloid.

MATHEMATICAL NOTES.

I. Let $$A^3 - 3AB = D^3:$$

and, consequently, $B = -\frac{1}{3A}(D^3 - A^3)$;

then the equation at line 3, p. 249 of the second volume of the former series of this Journal becomes

$$y = -\frac{1}{3A}\cdot\frac{D^3 - A^3}{D - A}.$$

The further reductions are obvious. By writing aD for D (where a is one of the values of $1^{\frac{1}{3}}$) we obtain the three values of y.

J. C.

Devereux Court, March 17, 1846.

II. If

$$\begin{aligned} A &= aa' - bb' - cc', & D &= bc' + cb', \\ B &= bb' - cc' - aa', & E &= ca' + ac', \\ C &= cc' - aa' - bb', & F &= ab' + ba'; \end{aligned}$$

then

$$ABC - AD^2 - BE^2 - CF^2 + 2DEF = (a^2 + b^2 + c^2)(aa' + bb' + cc')(a'^2 + b'^2 + c'^2).$$

Under the same conditions

$$(A + B)(B + C)(C + A) - 2DEF = (A + B)F^2 + (B + C)D^2 + (C + A)E^2.$$

H. (1).

III. *On the Equation of Payments.*—In the tenth No. of the *Cambridge Mathematical Journal* the Theory of the Equation of Payments is briefly considered. The method there followed, in the case of simple interest, leads to the result ordinarily employed, as an approximate solution; whereas it is in fact as much entitled to be considered exact as any solution obtained on the supposition of simple interest can be. The reason of this misapprehension has been overlooked in the above-mentioned paper, as well as in the common treatises on Algebra.

Suppose that two sums, s_1 and s_2, are due at two periods, t_1 and t_2 respectively; that it is required to find the period T at which the sum $s_1 + s_2$ may be paid without injury to either party, simple interest being allowed.

There are three methods of making the arrangement, which at first sight appear equally fair.

1. The time T ought to be such, that the present worth of s_1 due at t_1, together with the present worth of s_2 due at t_2, shall be equal to the present worth of $s_1 + s_2$ due at T.

2. It ought to be such, that the interest of s_1 from the time t_1 to the time T, shall be equal to the discount of s_2 for the interval between T and t_2.

3. Or lastly, it ought to be such that s_1, with its interest from t_1 to t_2, together with s_2, shall be equal to $s_1 + s_2$ with its interest during the interval between T and t_2.

If these methods are equally just, they ought to give the same value of T. Let us see whether they do so.

Take r for the rate of simple interest—supposed the same for each of the sums. Then, according to the usual formulæ, the first method will give rise to the equation

$$\frac{s_1}{1+rt_1}+\frac{s_2}{1+rt_2}=\frac{s_1+s_2}{1+rT};$$

whence $$T=\frac{s_1t_1+s_2t_2+rt_1t_2(s_1+s_2)}{s_1+s_2+r(s_1t_2+s_2t_1)}\;\ldots\ldots\ldots\ldots(1).$$

The second arrangement is, in symbolical language,

$$s_1(T-t_1)\,r=\frac{s_2r\,(t_2-T)}{1+r\,(t_2-T)},$$

or $T^2s_1r-T\{s_1+s_2+(t_1+t_2)\,rs_1\}+s_1t_1+s_2t_2+t_1t_2rs_1=0\ldots(2)$.

By the third method,

$$s_1\{1+r\,(t_2-t_1)\}+s_2=(s_1+s_2)\,\{1+r\,(t_2-T)\};$$

from which we have $$T=\frac{s_1t_1+s_2t_2}{s_1+s_2}\;\ldots\ldots\ldots\ldots\ldots(3).$$

It appears then that these three modes of determining T will not give the same result: and the reason of this we will explain.

But first it may be observed, that the results themselves would indicate that the third is the only correct one: for it is clear that the termination of the time T ought to be fixed with reference to the terminations of t_1 and t_2, independently of the epoch, which we choose to call *present*, and from which we calculate the *present worth;* in other words, t_2-T or $T-t_1$ ought to be independent of the absolute values of t_1 and t_2, and to depend only upon the interval t_2-t_1. This, it may be easily seen, is the case in (3), but not in (1) or (2).

The reason of the discrepancy will be found in our having made our calculations upon the supposition of *simple* interest, a supposition which implies that the interest is never added to the principal and made a part of it. But in our first method this condition has been violated; for instead of the actual case we have substituted the following imaginary one. If A receive $\frac{s_1}{1+rt_1}$ *now*, it is the same thing to him (*i.e.* he will be in the same position at any future time) as if he receive s_1 at the end of t_1. There is, however, this difference:—if he receive $\frac{s_1}{1+rt_1}$ now, he will have $\frac{s_1}{1+rt_1}+\frac{s_1rt_1}{1+rt_1}$

at the time t_1, but upon the former only of these quantities is he to receive interest from the time t_1, *because the interest is simple:* whereas, if he receive s_1 at the time t_1, he will have the same sum at that time as before, but will receive interest *on the whole* of it from the time t_1.

The imaginary case then which we have substituted for the actual one, and which we have expressed in equation (1), is not equivalent to it, and we cannot therefore rely upon the result.

For the same reason the result in the second case is not correct. It requires no further explanation than to say, that discount implies the calculation of the present worth, and so introduces an error similar to that in the first method. But in the third case, the creditor will be in the same position at the time t_2, and therefore at any subsequent time, as if the two separate payments were made.

If r be supposed so small that the terms of which it is a factor may be neglected, equations (1) and (2) will agree with (3).

Since we have shewn the disagreement of our results in the three cases to arise from the fact of simple interest being considered, it ought to disappear when the interest is compound. We should thus have, for the first case,

$$\frac{s_1}{(1+r)^{t_1}} + \frac{s_2}{(1+r)^{t_2}} = \frac{s_1+s_2}{(1+r)^T},$$

therefore $$s_1(1+r)^{t_2-t_1} + s_2 = (s_1+s_2)(1+r)^{t_2-T} \quad \ldots\ldots (4);$$

for the second,

$$s_1(1+r)^{T-t_1} - s_1 = s_2 - \frac{s_2}{(1+r)^{t_2-T}},$$

therefore $$s_1(1+r)^{t_2-t_1} + s_2 = (s_1+s_2)(1+r)^{t_2-T} \quad \ldots\ldots (5);$$

and for the third,

$$s_1(1+r)^{t_2-t_1} + s_2 = (s_1+s_2)(1+r)^{t_2-T} \quad \ldots\ldots (6);$$

and the three resulting equations now coincide.

H. Y.

END OF VOL. I.

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SCIENTIFIC JOURNALS.

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Tome XI. (1846) No. III. Note sur l'évaluation de l'aire de la surface nommée, dans l'optique, surface d'élasticité; par M. *William Roberts.*—Sur un théorème de M. Joachimsthal, relatif aux lignes de courbure planes; par *J. Liouville.*—Théorie géométrique de la lemniscate et des courbes elliptiques de la première classe; par M. *J. A. Serret.*—Sur l'équation

$$\frac{d^2y}{dt^2} = \frac{y}{(e^t + e^{-t})^2};$$

par M. *Besge.*—Extrait d'une Lettre adressée à M. Hermite; par M. *C. G. J. Jacobi.*—Construction des caustiques par réflexion sur les courbes planes, le point lumineux étant dans le plan de la courbe; par M. *J. H. Grillet.*—Nouvelles démonstrations des deux équations relatives aux tangentes communes à deux surfaces du second degré homofocales; Et propriétés des lignes géodésiques et des lignes de courbure de ces surfaces; par M. *Chasles.*—Notes sur quelques questions de priorité, au sujet d'un Mémoire de M. Mac Cullagh; par M. *Chasles.*—No. IV. Notes sur quelques questions de priorité, au sujet d'un Mémoire de M. Mac Cullagh; par M. *Chasles.* (Fin.)—Extrait d'une Lettre adressée à M. Liouville; par M. *William Roberts.*—Remarques sur les systèmes de droites dans l'espace; par M. *J. Bouquet.*—Note sur le théorème de M. Cauchy relatif au développement des fonctions en series; par M. *Ernest Lamarle.*—Expression numérique des intégrales définies qui se présentent, quand on cherche les termes généraux du développement des coordonnées d'une planète, dans son mouvement elliptique; par M. *F. Lefort.*—No. V. Note sur les centres des lignes et des surfaces algébriques; par M. *Breton (de Champ).*—Sur l'évaluation de quelques intégrales définies par des fonctions elliptiques; par M. *William Roberts.*—Note sur l'attraction; par M. *C. Briot.*—Sur l'interpolation; par M. *E. Brassinne.*—Sur les trajectoires qui coupent, sous un angle constant, les courbes méridiennes des surfaces de révolution; par M. *l'Abbé Aoust.*—No. VI. Note sur les équations d'équilibre d'un système de forces dirigées d'une manière quelconque dans l'espace; par M. *R. Lobatto.*—Sur les intégrales définies

$$\int_0^\infty \frac{e^{-\beta x} \,.\, x^{m-1}\, dx}{1 + x^2}, \quad \int_0^\infty \frac{\cos \beta x \,.\, x^{m-1}\, dx}{1 + x^2}, \quad \int_0^\infty \frac{\sin \beta x \,.\, x^{m-1}\, dx}{1 + x^2};$$

par M. *A. F. Svanberg.*—Note sur quelques intégrales multiples; par M. *William Roberts.*—Démonstration d'un théorème de Poisson; par M. *William Roberts.*—Note sur un problème de mécanique; par M. *E. Catalan.*—Note sur la propriété de la cycloïde, d'être la seule tautochrone dans le vide; par M. *E. L. Guillon.*—Lettres sur diverses questions d'analyse et de physique mathématique concernant l'ellipsoïde, adresseés à *M.* P. H. Blanchet; par *J. Liouville.* (Première Lettre.)—Extrait d'une Lettre adressée à *M.* J. Steiner, par M. *C. G. J. Jacobi.*

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BAND XXXII. (1846) Heft I.—1. Théorémes généraux sur les dérivées d'un ordre quelconque, de certaines fonctions très générales. Par M. *O. Schlömilch*, docteur es-sciences er lecteur des mathématiques à l'université de Jena.—2. Uber den Werth, welchen das bestimmte Integral $\int_0^{2\pi} \frac{d\phi}{1 - A \cos \phi - B \sin \phi}$ für beliebige imaginäre Werthe von A und B annimmt. Von Herrn Prof. Dr. *C. G. J. Jacobi* zu Berlin.—3. Mémoire sur les différentes manières de se servir de l'élasticité de l'air atmosphérique comme force motrice sur les chemins de fer. Une de ces manières constitue les chemins de fer *atmosphériques* proprement dits. Par l'éditeur.—4. Beiträge zur theorie der elliptischen functionen. Von Herrn Dr. phil. *G. Eisenstein* zu Berlin. (Fortsetzung der Abhandlung Nr. 14, Band 30, Heft 3.) III. Fernere Bemerkungen zu den transformationsformeln.—5. Notiz über partialbrüche. Von Herrn *G. Eisenstein*, Dr. phil. zu Berlin.—6. Uber Lehrsätze, von welchen die bekannten sätze über parallele curven besondere fälle sind. Von Herrn Prof. *J. Steiner* zu Berlin. (Auszug aus einer am 26 März in der hiesigen Akad. der Wiss. gehaltenen Vorlesung.) —7. Sur un moyen général de vérifier l'expression du potentiel relatif à une masse quelconque, homogène ou hétérogène. Par Mr. *G. Lejeune Dirichlet* à Berlin.—8. Uber die stabilität des gleichgewichts. Von Herrn Prof. *G. Lejeune Dirichlet* zu Berlin. (Auszug aus einer am 22 Januar 1846 in der Königl. Akademie der Wissenschaften gelesenen Abhandlung.)—9. Eine Bemerkung zur Zahlentheorie. Von Herrn Prof. Dr. *Stern* zu Göttingen.—10. Zur näherungsweisen Kreis-Quadratur.—Fac-simile einer auf der Konigl. Bibliothek zu Berlin befindlichen Handschrift von *Maupertuis*. Heft II. 11. Von denjenigen Moduln, welche Potenzen von Primzahlen sind. Von Herrn Oberlehrer Dr. *Schönemann* am Gymnasio zu Brandenburg an der Havel. (Fortsetzung und Schlufs der Abhandlung No. 22, im vorigen Bande.)—12. Variationum quas elementa motus perturbati planetarum subeunt nova et facilis evolutio. Auct. *Aug. Ferd. Möbius*, Prof. ord. Lipsiæ.—13. Sur quelques propriétés des déterminants gauches. Par Mr. *A. Cayley* de Cambridge.—14. Mémoire sur les différentes manières de se servir de l'élasticité de l'air atmosphérique comme force motrice sur les chemins de fer. Une de ces manières constitue les chemins de fer *atmosphériques* proprement dits. Par l'éditeur. (Suite du No. 3 du cahier précédant.)—15. Beweis des Satzes, dafs jede nicht fünfeckige Zahl eben so oft in eine gerade als ungerade Anzahl verschiedener Zahlen zerlegt werden kann. Von Herrn Prof. Dr. *C. G. J. Jacobi* zu Berlin.—16. Extrait d'une lettre adressée à M. *Hermite*. Par M. *C. G. J. Jacobi*.—17. Geometrische Lehrsätze. Von Herrn Prof. *J. Steiner* zu Berlin. (Auszug aus einer am 27 Nov. 1845, in der Akad. der Wissenschaften zu Berlin gehaltenen Vorlesung.)—Fac-simile einer von dem Herrn Geheimen Oberbergrath *Karsten* zu Berlin dem Herausgeber gefälligst mitgetheilten Handschrift von *W. J. G. Karsten*. Heft III. 18. Uber die Vertauschung von Parameter und Argument bei der dritten Gattung der Abelschen und höhern Transcendenten. Von Herrn Prof. Dr. *C. G. J. Jacobi* zu Berlin.—19. Uber einige der Binominalreihe analoge Reihen. Von Herrn Prof. Dr. *C. G. J. Jacobi* zu Berlin.—20. Verwandlung von Reihen in Kettenbrüche. Auszug eines Schreibens des Herrn Dr. *E. Heine*, Privatdocenten zu Bonn, an den Herrn Prof. *C. G. J. Jacobi* in Berlin.—21. Uber die Reihe

$$1 + \frac{(q^\alpha - 1)(q^\beta - 1)}{(q-1)(q^\gamma - 1)} x + \frac{(q^\alpha - 1)(q^{\alpha+1} - 1)(q^\beta - 1)(q^{\beta+1} - 1)}{(q-1)(q^2-1)(q^\gamma - 1)(q^{\gamma+1} - 1)} x^2 + \ldots$$

Aus einem Schreiben des Herrn Dr. *Heine* zu Bonn an den Herrn Prof. *Lejeune Dirichlet*.—22. Uber die Reduction des Integrals $\int \frac{fx\,dx}{\sqrt{\pm(1-x^8)}}$ auf elliptische

Integrale. Von Herrn Prof. *F. Richelot* zu Königsberg.—23. Beweis eines Satzes über elliptische Functionen. Von Herrn Prof. *F. Richelot* zu Königsberg.—24. Uber eine neue Methode zur Integration der hyperelliptischen Differentialgleichungen und über die rationale Form ihrer vollständigen algebraischen Integralgleichungen. Von Herrn Prof. Dr. *C. G. J. Jacobi* zu Berlin.—25. Neue Eigenschaft zweier Kräfte, durch welche ein Kräftensystem ersetzt werden kann. Von dem Herrn Geh. Hofrath und Professor Dr. *Schweins* zu Heidelberg.—26. Mémoire sur les différentes manières de se servir de l'élasticité de l'air atmosphérique comme force motrice sur les chemins de fer. Une de ces manières constitue les chemins de fer *atmosphériques* proprement dits. Par l'éditeur. (Suite du No. 3 au premier et No. 14 au second cahier de ce tome.)—Fac-simile einer von dem Herrn Professor Dr. *Stern* zu Göttingen dem Herausgeber gefälligst mitgetheilten Handschrift von *Schröter*.

Band XXXIII. (1846) Heft. I.—1. Untersuchungen über die analytischen Facultäten. Von dem Herrn Prof. *Ottinger* zu Freiburg im Br.—2. Uber periodische Kettenbrüche. Von Herrn Dr. *H. Siebeck* zu Breslau.—3. Die recurrenten Reihen, vom Standpuncte der Zahlentheorie aus betrachtet. Von Herrn Dr. *H. Siebeck* zu Breslau.—4. Uber independente Darstellung der höheren Differentialquotienten und den Gebrauch des Summenzeichens. Von Herrn Cand. math. *R. Hoppe* zu Berlin.—5. Nota sopra l'equazione di una curva del sesto ordine, che s'incontra in un problema riguardante l'ellissi. Del Signor *Barn. Tortolini* in Roma. (Estratta dalla raccolta scientifica num. 6, an. 11.)—Fac-simile einer von dem Herrn Professor *Möbius* zu Leipzig dem Herausgeber gefälligst mitgetheilten Handschrift von *Réaumur*.—Heft. II. 6. Uber die Bedingung der Integrabilität. Von Herrn Dr. *F. Jochimsthal*, Privatdocenten a. d. Universität zu Berlin. (Abgedruckt aus dem Osterprogramm 1844 der Königl. Realschule zu Berlin.)—7. Untersuchungen über die analytischen Facultäten. Von dem Herrn Prof. *Ottinger* zu Freyburg im Br. (Fortsetzung der Abhandlung No. 1, im vorigen Heft.)—8. Einiges über die Berechnung der reellen Wurzeln numerischer Gleichungen mittels ohne Ende fortlaufender Reihen. (Von dem Herrn Dr. *Waltinowsky* in Triest.) —9. Uber die Verwandlung der Reihen in Kettenbrüche. (Von Herrn Dr. *Heilermann* in Cöln a. R.—Fac-simile einer von dem Herrn Professor *Möbius* zu Leipzig dem Herausgeber gefälligst mitgetheilten Handschrift von *Méchain*.